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Cramér-Rao Bound, MUSIC, And Maximum Likelihood

Effects of Temporal Phase Difference

C. V. Tran

NAVAL OCEAN SYSTEMS CENTER
San Diego, California 92152-5000

J. D. FONTANA, CAPT, USN
Commander

H. R. TALKINGTON, Acting
Technical Director

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SUMMARY

We derived closed-form expressions for the Cramér-Rao Bound, MUSIC, and Maximum Likelihood (ML) asymptotic variances corresponding to the two-source direction-of-arrival estimation where sources were modeled as deterministic signals impinging on a uniform linear array. The choice of the center of the array as the coordinate reference resulted in compact expressions that greatly facilitated our study of effects of temporal phase difference (correlation phase) of the two sources on asymptotic variances of estimation error. These effects were shown to be intensified when the two signals were closely spaced and/or when their normalized correlation magnitude was high. Numerical examples obtained from specializing general formulas agreed with our results.

KEY SYMBOLS

m = number of sensors

n = number of arrivals

N = number of snapshots

$\mathbf{a}_j(\omega_j), \quad 1 \leq j \leq n \quad : m \times 1$ steering vectors

$\mathbf{d}_j = d\mathbf{a}_j(\omega_j)/d\omega_j, \quad 1 \leq j \leq n$

$\mathbf{\Omega} = (\omega_1, \omega_2, \ldots, \omega_n)^T$: parameter vector

$\mathbf{A} = \left[\mathbf{a}_1(\omega_1) \vdots \cdots \vdots \mathbf{a}_n(\omega_n)\right]$: $m \times n$ transfer matrix (direction matrix)

$\mathbf{D} = \left[\mathbf{d}_1 \vdots \cdots \vdots \mathbf{d}_n\right]$

$\mathbf{A}^+ = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*$: left pseudo-inverse of $\mathbf{A}$

$\mathbf{P_A} = \mathbf{AA}^+$: orthogonal projection on subspace spanned by columns of $\mathbf{A}$

$\mathbf{P}_\mathbf{A}^\perp = \mathbf{I} - \mathbf{P_A}$

$\mathbf{H} = \mathbf{D}^*\mathbf{P}_\mathbf{A}^\perp\mathbf{D}$

$\mathbf{S} = n \times n$ source covariance matrix

$\mathbf{R} = \mathbf{ASA}^* + \sigma^2\mathbf{I} = \mathbf{E}_s\mathbf{\Lambda}_s\mathbf{E}_s^* + \mathbf{E}_n\mathbf{\Lambda}_n\mathbf{E}_n^*$: $m \times m$ covariance matrix of array output

$\lambda_1 \geq \cdots \geq \lambda_d > \lambda_{d+1} = \cdots = \lambda_m = \sigma^2, \quad d \leq n$: eigenvalues of $\mathbf{R}$

$\mathbf{E}_s = \left[\mathbf{e}_1 \vdots \cdots \vdots \mathbf{e}_d\right]$: columns of $\mathbf{E}_s$ are normalized eigenvectors corresponding to the d largest eigenvalues of $\mathbf{R}$

$\mathbf{E}_n = \left[\mathbf{e}_{d+1} \vdots \cdots \vdots \mathbf{e}_m\right]$: columns of $\mathbf{E}_n$ are normalized eigenvectors corresponding to the eigenvalue σ^2 of multiplicity $m - d$

$$\mathbf{\Lambda}_n = \sigma^2\mathbf{I}, \qquad \mathbf{\Lambda}_s = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_d \end{pmatrix}$$

$\mathbf{M}^{-T} = (\mathbf{M}^{-1})^T = (\mathbf{M}^T)^{-1}$

$\mathbf{M} \odot \mathbf{N}$: Schur (Hadamard, elementwise) product, $\quad (\mathbf{M} \odot \mathbf{N})_{ij} = \mathbf{M}_{ij}\mathbf{N}_{ij}$

CONTENTS

FIGURES

1. INTRODUCTION

It is well recognized (references [1], [2], [3], [4], [5], [6], [7]) that, for many important signal processing applications, the pertinent problem is estimation of the parameters of the following basic model:

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \tag{1}$$

where $\mathbf{x}(t)$ is an observed data vector of size m, with m equal to the number of sensors, the n-vector $\mathbf{s}(t)$ contains the complex envelopes of the n narrowband signals from far-field emitters, $\mathbf{n}(t)$ denotes a complex m-vector of additive noise, and the columns of the $m \times n$ transfer (direction) matrix $\mathbf{A}(\Omega)$ are steering vectors $\mathbf{a}_j(\omega_j)$, $1 \leq j \leq n$, with the unknown parameter vector $\boldsymbol{\Omega} = (\omega_1, \omega_2, \ldots, \omega_n)^T$ corresponding to the unknown directions of arrivals (DOAs).

The waveforms $\mathbf{s}(t)$ are modeled as stationary, jointly Gaussian processes with zero mean and second moments

$$\mathrm{E}\left(\mathbf{s}(t)\mathbf{s}^*(s)\right) = \mathbf{S}\delta_{t,s}, \qquad \mathrm{E}\left(\mathbf{s}(t)\mathbf{s}^T(s)\right) = \mathbf{0},$$

where, as usual, the operator $E(\cdot)$ denotes the expected value, the superscript T for transpose, the superscript $*$ for complex conjugate and $\delta_{t,s}$ is the Kronecker delta.

Similarly, the additive noise $\mathbf{n}(t)$, uncorrelated with the signal waveforms, is assumed to be a stationary, zero mean, Gaussian process with second moments

$$\mathrm{E}\left(\mathbf{n}(t)\mathbf{n}^*(s)\right) = \sigma^2\mathbf{I}\delta_{t,s}, \qquad \mathrm{E}\left(\mathbf{n}(t)\mathbf{n}^T(s)\right) = \mathbf{0}.$$

The array output, being observed at N discrete time instances, is a stationary, zero mean Gaussian process with second moments

$$\mathrm{E}\left(\mathbf{x}(t)\mathbf{x}^*(s)\right) = \mathbf{R}\delta_{t,s} = \{\mathbf{A}(\boldsymbol{\Omega})\mathbf{S}\mathbf{A}^*(\boldsymbol{\Omega}) + \sigma^2\mathbf{I}\}\delta_{t,s}, \qquad \mathrm{E}\left(\mathbf{x}(t)\mathbf{x}^T(s)\right) = \mathbf{0}.$$

It is well known that $\mathbf{R}$ has the following eigen-decomposition

$$\mathbf{R} = \sum_{j=1}^{m} \lambda_j \mathbf{e}_j \mathbf{e}_j^* = \mathbf{E}_s\boldsymbol{\Lambda}_s\mathbf{E}_s^* + \mathbf{E}_n\boldsymbol{\Lambda}_n\mathbf{E}_n^*,$$

with

$$\lambda_1 \geq \cdots \geq \lambda_d > \lambda_{d+1} = \cdots = \lambda_m = \sigma^2, \quad d \leq n,$$

where the columns of $\mathbf{E}_s$ are the eigenvectors of the corresponding d largest eigenvalues and the columns of $\mathbf{E}_n$ are the $n-d$ eigenvectors of the eigenvalue σ^2. Here, the diagonal matrix

$\mathbf{\Lambda}_s$ contains the d largest, in descending order, eigenvalues of $\mathbf{R}$. The range of $\mathbf{E}_s$ is often referred to as the signal subspace and its orthogonal complement, which is spanned by $\mathbf{E}_n$, as the noise subspace.

The problem of determination of the number of arrivals n belongs to the area of detection and will not be discussed here. It is assumed that n and d are known and one wishes to estimate the DOA vector $\mathbf{\Omega}$.

Recently there has been interest in treating the matrix S other than real for two-source applications (references [8], [9], [10], [11], [12]). Earlier analytical results in this direction for the Cramér-Rao bound can be found in references [13], [14], [15], and most recently in reference [16]. The purpose of this report is to investigate effects of temporal phase difference (that is, correlation phase) of two sources on asymptotic variances of estimation error using a uniform linear array. In particular, we shall restrict to the Cramér-Rao bound (CRB), the MUSIC and Maximum Likelihood (ML) methods that are of practical and theorectical interest (references [2], [3], [4]). Our results also provide an analytical background for simulations such as those performed in references [10] and [12].

This paper is organized as follows. In section 2, we shall collect several important asymptotic results on estimation error for the deterministic-signal model, i.e., in (1) the waveforms $\mathbf{s}(t)$ are fixed in all realizations of the random data $\mathbf{x}(t)$. Among these are the CRB for the estimation of the parameter vector $\mathbf{\Omega}$ of (1), the asymptotic covariance matrix for the MUSIC method and that for the ML method – all three were derived by Stoica and Nehorai (references [2], [3]). Recently, the asymptotic covariance matrix for a general multidimensional signal subspace method, which includes ML, has been obtained by Viberg and Ottersten (references [5], [6], [7]). For an appropriate choice of the weighting matrix, the resulting covariance matrix of estimation error is smallest among those in the weighted subspace fitting (WSF).

We shall specify the geometry of the sensor array and the number of arrivals in section 3. Precisely, we shall study the situation of two narrowband plane waves impinging on a uniform linear array (ULA). This will specify the form of the steering vectors, and thus the transfer matrix. Basic properties of ULA that are essential for subsequent development are derived. Some of these properties are independent of coordinate reference; others are not, and an inappropriate choice will result in unnecessarily complicated expressions and analysis. A suitable coordinate reference for our purpose is the center of the array.

In section 4, we shall be concerned with effects of the temporal phase difference on the CRB covariance. The main result in this section states that, for the coordinate reference at the center of the array, then the CRB variance of each estimation, considered as a function of the phase difference θ, is periodic of period 180^0, symmetric about 90^0, and on the interval where $0^0 \leq \theta \leq 180^0$, it is decreasing when $0^0 < \theta < 90^0$, increasing when $90^0 < \theta < 180^0$, and it assumes the minimum at 90^0 and maximum (either finite or infinite) at 0^0 and 180^0 (that is, either the two signals are in phase or out of phase). This behavior, observed in simulation studies (references [10], [12]), can be analytically deduced from reference [15]. Our proof follows the development in reference [2].

Results on effects of the temporal phase difference on asymptotic MUSIC covariance is derived in section 5. We analytically show that, for the coordinate reference at the center of the array, the asymptotic MUSIC variance of each estimation is of the form $A + B\cos(\theta)$, where $A > |B| \geq 0$ with B having the sign of the function $\wp(\Delta\omega; m) \triangleq \frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})}$. (The electrical angular separation $\Delta\omega$ will be defined subsequently in section 2 below). The behavior of the asymptotic MUSIC variance with respect to θ thus can be obtained immediately. In particular, suppose the electrical angular separation is within one beamwidth ($0 < \Delta\omega < \frac{2\pi}{m}$), then the asymptotic MUSIC variance is symmetric about 180^0, decreasing when $0^0 < \theta < 180^0$, increasing when $180^0 < \theta < 360^0$, attains its maximum at $\theta = 0^0$ (that is, the two signals are in phase) and minimum at $\theta = 180^0$ (that is, the two signals are out of phase).

In section 6, we examine effects of the temporal phase difference on the asymptotic ML variance, which is derived in terms of CRB variances and covariance. As in the case of the asymptotic MUSIC variance, the behavior of the asymptotic ML variance, as a function of θ, is also influenced by $\Delta\omega$ in terms of beamwidths. In particular, suppose the electrical angular separation is within one beamwidth ($0 < \Delta\omega < \frac{2\pi}{m}$), then the asymptotic ML variance is symmetric about 180^0, and on the interval where $0^0 \leq \theta \leq 180^0$, it attains the maximum at $\theta = 0^0$ (that is, the two signals are in phase) and minimum at $\theta = \theta_0$ with $90^0 < \theta_0 \leq 180^0$, where the exact location of θ_0 depends on $\Delta\omega$ and other parameters.

Numerical results obtained from specializing general formulas surveyed in section 2 are presented in section 7. These numerical results are consistent with our analytical results. As discussed in sections 4 to 6, effects of temporal phase difference on CRB, asymptotic MUSIC, and ML variances of estimation error are intensified when the two sources are closely spaced and/or when their normalized correlation magnitude is high. This behavior is well noticeable in figures 1(a) through 3(d) .

Finally, in section 8, we conclude the paper by summarizing our findings with emphasis on the case where the two arrivals are closely spaced with their electrical angular separation within one beamwidth.

2. ASYMPTOTIC COVARIANCE MATRICES OF ESTIMATION ERROR

The Cramér-Rao bound (**CRB**) provides a lower bound for the covariance matrix of the estimation error of any unbiased estimate. Stoica and Nehorai (reference [4]) gave a general formula for the **CRB** from which they derived expressions for the case where signals are deterministic as well as the case where signals are random. Since the expression of **CRB** for random signals is much more complicated than its deterministic counterpart, and since deterministic **CRB** is the lower bound of random **CRB** (reference [4]), hereafter we shall restrict our attention to the deterministic-signal model, i.e., in (1) the waveforms $\mathbf{s}(t)$ are fixed in all realizations of the random data $\mathbf{x}(t)$ and only $\mathbf{n}(t)$ varies from realization to realization. For deterministic signals, Stoica and Nehorai proved

Theorem 1 (reference [2]) *Let $\hat{\mathbf{\Omega}}_N$ be an (asymptotic) unbiased estimate of the true parameter vector $\mathbf{\Omega}$. For large N, the Cramér-Rao inequality can be written*

$$N\mathrm{E}\left((\hat{\mathbf{\Omega}}_N - \mathbf{\Omega})(\hat{\mathbf{\Omega}}_N - \mathbf{\Omega})^T\right) \geq \mathbf{CRB},$$

where

$$\mathbf{CRB} = \frac{\sigma^2}{2}\left[\mathrm{Re}(\mathbf{H} \odot \mathbf{S}^T)\right]^{-1}. \tag{2}$$

Here the matrices $\mathbf{D}$ *and* $\mathbf{P}_{\mathbf{A}}^{\perp}$ *are evaluated at* $\mathbf{\Omega}$.

Hereafter, we shall refer to

$$\lim_{N\to\infty} N\mathrm{E}\left((\hat{\mathbf{\Omega}}_N - \mathbf{\Omega})(\hat{\mathbf{\Omega}}_N - \mathbf{\Omega})^T\right) \tag{3}$$

as the asymptotic covariance matrix of the estimation error corresponding to the method that yields the estimate $\hat{\Omega}$ of the true parameter vector Ω. The matrix $\frac{2}{\sigma^2}\,\mathrm{Re}(\mathbf{H} \odot \mathbf{S}^T)$ is called the Fisher information matrix.

The asymptotic covariance matrix for the MUSIC method was also obtained by Stoica and Nehorai; in matrix form it reads

Proposition 1 (references [2], [3]) *Suppose the source covariance matrix* S *is of full rank. Then the asymptotic covariance matrix of the estimation error of the* MUSIC *method is*

$$\mathbf{C}_{MU} = \frac{\sigma^2}{2}(\mathbf{H} \odot \mathbf{I})^{-1}\mathrm{Re}(\mathbf{H} \odot \mathbf{K}^T)(\mathbf{H} \odot \mathbf{I})^{-1}, \tag{4}$$

where

$$\mathbf{K} = \mathbf{S}^{-1} + \sigma^2\mathbf{S}^{-1}(\mathbf{A}^*\mathbf{A})^{-1}\mathbf{S}^{-1}. \tag{5}$$

Notice that $(\mathbf{H} \odot \mathbf{I})^{-1}$ is a fancy expression of the matrix

$$\begin{bmatrix} \frac{1}{\mathbf{H}_{11}} & & & \\ & \frac{1}{\mathbf{H}_{22}} & & \\ & & \ddots & \\ & & & \frac{1}{\mathbf{H}_{nn}} \end{bmatrix}.$$

Recently, Viberg and Ottersten (references [5], [6], and [7]) have introduced a unified approach to multidimensional signal subspace method called weighted subspace fitting (WSF) by incorporating a general Hermitian weighting matrix $\mathbf{W}$ and the estimation error covariance is minimized with respect to this weighting. When $\mathbf{W} = \mathbf{I}$, the corresponding method is called the multidemensional MUSIC (MD-MUSIC), as opposed to the conventional (one-dimensional search) MUSIC.

The maximum likelihood (ML) method is known (references [5], [6]) to be equivalent to the weighted subspace fitting method with the weighting matrix $\mathbf{W} = \tilde{\mathbf{\Lambda}}$ where

$$\tilde{\mathbf{\Lambda}} \triangleq \mathbf{\Lambda}_s - \sigma^2\mathbf{I} = \begin{bmatrix} \lambda_1 - \sigma^2 & & & \\ & \lambda_2 - \sigma^2 & & \\ & & \ddots & \\ & & & \lambda_d - \sigma^2 \end{bmatrix}.$$

The asymptotic covariance matrix of the estimation error of the subspace fitting method with the weighting matrix $\mathbf{W}$ is given by

Theorem 2 (reference [6]) *For the subspace fitting method with the Hermitian positive definite weighting matrix* $\mathbf{W}$, *the asymptotic distribution of the estimation error is given by*

$$\sqrt{N}(\hat{\mathbf{\Omega}}_N - \mathbf{\Omega}) \sim \mathrm{N}(\mathbf{0}, \mathbf{C}_{WSF}),$$

where

$$\mathbf{C}_{WSF} = (\bar{\mathbf{V}}'')^{-1}\mathbf{Q}(\bar{\mathbf{V}}'')^{-1} \tag{6}$$

with

$$\bar{\mathbf{V}}'' = -2\mathrm{Re}\{\mathbf{H} \odot (\mathbf{A}^{+}\mathbf{E}_s\mathbf{W}\mathbf{E}_s^{*}\mathbf{A}^{+*})^T\}, \tag{7}$$

$$\mathbf{Q} = 2\sigma^2\mathrm{Re}\{\mathbf{H} \odot (\mathbf{A}^{+}\mathbf{E}_s\mathbf{W}\mathbf{\Lambda}_s\tilde{\mathbf{\Lambda}}^{-2}\mathbf{W}\mathbf{E}_s^{*}\mathbf{A}^{+*})^T\}. \tag{8}$$

Here, all expressions are evaluated at the true parameter $\mathbf{\Omega}$. The result holds for arbitrary signal correlation including full coherence.

When S has full rank, a convenient expression for $\mathbf{C}_{ML}$ can be derived from (6) to (8):

Corollary 1 (reference [6]) *Assume the source covariance matrix* S *is nonsingular. Then the asymptotic covariance matrix of the estimation error of the* ML *method* ($\mathbf{W} = \tilde{\mathbf{\Lambda}}$) *is*

$$\mathbf{C}_{ML} = \mathbf{CRB}\left[\mathbf{I} + 2\mathrm{Re}\{\mathbf{H} \odot (\mathbf{A}^{*}\mathbf{A})^{-T}\}\mathbf{CRB}\right]. \tag{9}$$

This expression is consistent with an earlier expression obtained by Stoica and Nehorai in reference [3].

3. BASIC PROPERTIES OF UNIFORM LINEAR ARRAY

In this section, we shall discuss fundamental properties of a ULA that are pertinent to subsequent development. Some properties are independent of the choice of coordinate reference (Lemmas 1 and 2); others are not, and an unsuitable choice will result in unnecessarily complicated expressions and analysis. An appropriate coordinate reference for our purpose is the center of the array, which will become clear by the end of the section.

We shall consider the situation of two narrowband plane signals impinging on a uniformly distributed linear array. In this case, the parameter vector is $\mathbf{\Omega} = (\omega_1, \omega_2)^T$. In view of reference [17], we shall assume the number of array elements $m > 3$.

Let $\Delta\omega = |\omega_1 - \omega_2|$ be the electrical angular separation of the two signals that will be defined shortly. In this section, we shall prove that the asymptotic covariance matrix **CRB**, $\mathbf{C}_{MU}$, $\mathbf{C}_{WSF}$ (and thus $\mathbf{C}_{ML}$ and $\mathbf{C}_{W_0}$) are dependent on the electrical angular separation $\Delta\omega$ and independent of the absolute values ω_1 and ω_2. An analogous temporal result for **CRB** was shown in references [13] and [14]. Proceeding from spatial premises of the basic mod‹' stated in (1), our development gives a direct treatment and insight to the two-source directicn finding (DF) using a uniform linear array. Consider the general case where the coordinate origin $\bigcirc$ is at a point on the line segment connecting the first sensor and the last (m-th) sensor. In this setting, if the coordinate origin $\bigcirc$ is between the p-th sensor and the $(p+1)$-st sensor, for some $1 \leq p \leq m-1$, and let

$$r \triangleq \frac{\text{distance between } \bigcirc \text{ and } p\text{-th element}}{\text{interelement spacing}}, \qquad \text{where } 0 \leq r \leq 1, \tag{10}$$

then the steering vector $\mathbf{a}_j(\omega_j)$, for $j = 1, 2$, is given by

$$\mathbf{a}_j(\omega_j) = \left[e^{-\mathrm{i}(p-1+r)\omega_j}, \ldots, e^{-\mathrm{i}(1+r)\omega_j}, e^{-\mathrm{i}r\omega_j}, e^{\mathrm{i}(1-r)\omega_j}, \ldots, e^{\mathrm{i}(m-p-r)\omega_j}\right]^T$$

where

$$\omega_j = \frac{2\pi d}{\lambda} \sin(\phi_j)$$

denotes the electrical phase angle corresponding to the angle of arrival ϕ_j, measured with respect to the normal to the array, of the j-th plane wave with d as the interelement spacing and λ as the array wavelength.

The case where the coordinate center is chosen to be at a particular sensor element can be obtained from the general formulation by setting either $r = 0$ or $r = 1$. A convenient choice of coordinate origin of the array is its "center," which is its actual middle sensor if m, the number of array elements, is odd, otherwise it is a fictitious center. When m is even, the choice of the fictitious center as the coordinate reference corresponds to setting $p = \frac{m}{2}$ and $r = \frac{1}{2}$ in the general case.

For a one-to-one correspondence between the values of ϕ_j and ω_j, we shall assume that $d \leq \lambda/2$ (reference [18], page 27). Thus, by reorder'ng the two plane waves if necessary, without loss of generality, we can always write the direction matrix as

$$\mathbf{A}(\text{origin at } \bigcirc) = \begin{bmatrix} e^{-\mathrm{i}(p-1+r)\omega} & e^{-\mathrm{i}(p-1+r)(\omega+\Delta\omega)} \\ \vdots & \vdots \\ e^{-\mathrm{i}(1+r)\omega} & e^{-\mathrm{i}(1+r)(\omega+\Delta\omega)} \\ e^{-\mathrm{i}r\omega} & e^{-\mathrm{i}r(\omega+\Delta\omega)} \\ e^{\mathrm{i}(1-r)\omega} & e^{\mathrm{i}(1-r)(\omega+\Delta\omega)} \\ \vdots & \vdots \\ e^{\mathrm{i}(m-p-r)\omega} & e^{\mathrm{i}(m-p-r)(\omega+\Delta\omega)} \end{bmatrix}, \tag{11}$$

where ω and $\omega + \Delta\omega$ are electrical phase angles with $\Delta\omega \leq \pi$. Consequently,

$$\mathbf{A}(\text{origin at } \bigcirc) = e^{-\mathrm{i}(p-1+r)\omega}\, \mathbf{U}\, \mathbf{A}_0(\text{origin at } \bigcirc), \tag{12}$$

where

$$\mathbf{U} = \begin{bmatrix} 1 & & & & \\ & e^{\mathrm{i}\omega} & & & \\ & & e^{\mathrm{i}2\omega} & & \\ & & & \ddots & \\ & & & & e^{\mathrm{i}(m-1)\omega} \end{bmatrix}, \quad \mathbf{A}_0(\text{origin at } \bigcirc) = \begin{bmatrix} 1 & e^{-\mathrm{i}(p-1+r)\Delta\omega} \\ \vdots & \vdots \\ 1 & e^{-\mathrm{i}(1+r)\Delta\omega} \\ 1 & e^{-\mathrm{i}r\Delta\omega} \\ 1 & e^{\mathrm{i}(1-r)\Delta\omega} \\ \vdots & \vdots \\ 1 & e^{\mathrm{i}(m-p-r)\Delta\omega} \end{bmatrix}. \tag{13}$$

Note that $\mathbf{U}$ is unitary, that is $\mathbf{U}\mathbf{U}^* = \mathbf{I} = \mathbf{U}^*\mathbf{U}$.

In the same manner,

$$\mathbf{D}(\text{origin at } \bigcirc) = \begin{bmatrix} -\mathrm{i}(p-1+r)e^{-\mathrm{i}(p-1+r)\omega} & -\mathrm{i}(p-1+r)e^{-\mathrm{i}(p-1+r)(\omega+\Delta\omega)} \\ \vdots & \vdots \\ -\mathrm{i}(1+r)e^{-\mathrm{i}(1+r)\omega} & -\mathrm{i}(1+r)e^{-\mathrm{i}(1+r)(\omega+\Delta\omega)} \\ -\mathrm{i}re^{-\mathrm{i}r\omega} & -\mathrm{i}re^{-\mathrm{i}r(\omega+\Delta\omega)} \\ \mathrm{i}(1-r)e^{\mathrm{i}(1-r)\omega} & \mathrm{i}(1-r)e^{\mathrm{i}(1-r)(\omega+\Delta\omega)} \\ \vdots & \vdots \\ \mathrm{i}(m-p-r)e^{\mathrm{i}(m-p-r)\omega} & \mathrm{i}(m-p-r)e^{\mathrm{i}(m-p-r)(\omega+\Delta\omega)} \end{bmatrix} \tag{14}$$

can be factored as

$$\mathbf{D}(\text{origin at } \bigcirc) = e^{-\mathrm{i}(p-1+r)\omega}\, \mathbf{U}\, \mathbf{D}_0(\text{origin at } \bigcirc), \tag{15}$$

where

$$\mathbf{D}_0(\text{origin at } \bigcirc) = \begin{bmatrix} -(p-1+r)\mathrm{i} & -(p-1+r)\mathrm{i}e^{-\mathrm{i}(p-1+r)\Delta\omega} \\ \vdots & \vdots \\ -\mathrm{i}(1+r) & -\mathrm{i}(1+r)e^{-\mathrm{i}(1+r)\Delta\omega} \\ -\mathrm{i}r & -\mathrm{i}re^{-\mathrm{i}r\Delta\omega} \\ \mathrm{i}(1-r) & \mathrm{i}(1-r)e^{\mathrm{i}(1-r)\Delta\omega} \\ \vdots & \vdots \\ (m-p-r)\mathrm{i} & (m-p-r)\mathrm{i}e^{\mathrm{i}(m-p-r)\Delta\omega} \end{bmatrix}. \tag{16}$$

Using expressions (12) to (13) and (15) to (16) together with the fact that $\mathbf{U}$ is unitary, we see immediately

$$(\mathbf{A}^*\mathbf{A})^{-1} = (\mathbf{A}_0^*\mathbf{A}_0)^{-1}, \tag{17}$$

$$\mathbf{A}^+ = \mathbf{A}_0^+\mathbf{U}^*, \tag{18}$$

$$\mathbf{P}_{\mathbf{A}} = \mathbf{U}\mathbf{P}_{\mathbf{A}_0}\mathbf{U}^*, \tag{19}$$

$$\mathbf{P}_{\mathbf{A}}^{\perp} = \mathbf{U}\mathbf{P}_{\mathbf{A}_0}^{\perp}\mathbf{U}^*, \tag{20}$$

$$\mathbf{H} = \mathbf{D}_0^*\mathbf{P}_{\mathbf{A}_0}^{\perp}\mathbf{D}_0, \tag{21}$$

$$\mathbf{R} = \mathbf{U}\mathbf{R}_0\mathbf{U}^* \tag{22}$$

where $\mathbf{R}_0 \triangleq \mathbf{A}_0\mathbf{S}\mathbf{A}_0^* + \sigma^2\mathbf{I}$.

The expression (22) states that $\mathbf{R}$ and $\mathbf{R}_0$ are similar. It is well known that similar matrices have the same eigenvalues with the same multiplicities. Also, from (22), it follows that if $\mathbf{f}$ is an eigenvector of $\mathbf{R}_0$ then $\mathbf{Uf}$ is an eigenvector of $\mathbf{R}$ corresponding to the same eigenvalue, and if $\mathbf{e}$ is an eigenvector of $\mathbf{R}$ then $\mathbf{U}^*\mathbf{e}$ is an eigenvector of $\mathbf{R}_0$ corresponding to the same eigenvalue. Thus,

$$\mathbf{E}_s = \mathbf{U}\mathbf{F}_s \tag{23}$$

where

$$\mathbf{F}_s = \left[\mathbf{f}_1 \vdots \cdots \vdots \mathbf{f}_d\right]$$

with the columns $\mathbf{f}_1, \cdots, \mathbf{f}_d$ of $\mathbf{F}_s$ are normalized eigenvectors corresponding to eigenvalues $\lambda_1, \ldots, \lambda_d$ of $\mathbf{R}_0$ (also of $\mathbf{R}$). Apply (18), (21) and (23) to (7) and (8) we have

$$\bar{\mathbf{V}}'' = -2\mathrm{Re}\{\mathbf{H} \odot (\mathbf{A}_0^+\mathbf{F}_s\mathbf{W}\mathbf{F}_s^*\mathbf{A}_0^{+*})^T\} \tag{24}$$

$$\mathbf{Q} = 2\sigma^2\mathrm{Re}\{\mathbf{H} \odot (\mathbf{A}_0^+\mathbf{F}_s\mathbf{W}\mathbf{\Lambda}_s\tilde{\mathbf{\Lambda}}^{-2}\mathbf{W}\mathbf{F}_s^*\mathbf{A}_0^{+*})^T\}. \tag{25}$$

That is, in the case of two narrowband plane signals impinging on a uniformly spaced linear array, the asymptotic covariance matrix $\mathbf{C}_{WSF}$ of the estimation error using subspace fitting with the weighting matrix $\mathbf{W}$ in Theorem 2 is dependent on $\Delta\omega$ and independent of ω.

For two arrivals with signal powers $\pi_1, \pi_2 > 0$, the source covariance matrix $\mathbf{S}$ is given by

$$\mathbf{S} = \begin{bmatrix} \pi_1 & \sqrt{\pi_1\pi_2}\,\rho \\ \sqrt{\pi_1\pi_2}\,\bar{\rho} & \pi_2 \end{bmatrix} = \sqrt{\pi_1\pi_2}\begin{bmatrix} \sqrt{\frac{\pi_1}{\pi_2}} & |\rho|e^{i\theta} \\ |\rho|e^{-i\theta} & \sqrt{\frac{\pi_2}{\pi_1}} \end{bmatrix}, \tag{26}$$

where ρ is the normalized correlation between the first and second baseband signals and θ=arg(ρ) is the temporal phase difference. If ρ=0, the two signals are incoherent, $|\rho|$=1, they are coherent, and $0 < |\rho| < 1$, they are partially correlated. Since $e^{i\theta} = e^{i(\theta+2k\pi)}$ for any integer k, we shall consider only $0 \leq \theta \leq 2\pi$. Note that θ depends on, as being measured with respect to, the coordinate reference of the array.

Apply (17) and (21) to (2) to (5) and we see readily that **CRB** and $\mathbf{C}_{MU}$ are dependent on $\Delta\omega$ and independent of ω.

We summarize the above discussion in

Lemma 1 *For any two distinct narrowband plane signals with electrical angles ω_1 and ω_2 impinging on a uniform linear array, for any choice of coordinate origin along the line segment connecting the first and last sensing elements, the asymptotic covariance matrices* **CRB**, $\mathbf{C}_{MU}$, $\mathbf{C}_{WSF}$ *(thus, in particular,* $\mathbf{C}_{ML}$ *) are dependent on $\Delta\omega=|\omega_1 - \omega_2|$ and independent of ω_1 and ω_2. Furthermore, the result holds for arbitrary signal correlation including full coherence, except for* $\mathbf{C}_{MU}$.

Since $e^{\pm i(\theta+\pi)} = -e^{\pm i\theta}$, by applying (26) to (2), we obtain immediately the following result which can be deduced from references [13], [14].

Lemma 2 *For any coordinate reference of the array along the line connecting the first and last sensors, the elements of the 2×2 real symmetric matrix* **CRB**, *considered as functions of the temporal phase difference θ, satisfy*

$$[\mathbf{CRB}(\theta+\pi)]_{ii} = [\mathbf{CRB}(\theta)]_{ii} \quad \text{for } i = 1,2 \tag{27}$$

$$[\mathbf{CRB}(\theta+\pi)]_{12} = -[\mathbf{CRB}(\theta)]_{12}. \tag{28}$$

The matrix **H** is a prominent component of **CRB**, $\mathbf{C}_{MU}$, and $\mathbf{C}_{WSF}$ as seen in (2), (4), (7) and (8), as well as in (9). The next result, whose proof appears in Appendix A, gives a general description of **H**.

Lemma 3 *For any choice of coordinate origin on the line segment connecting the first and last array elements,* $\mathbf{H}$ *is of the form*

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \overline{\mathbf{H}}_{12} & \mathbf{H}_{11} \end{bmatrix}, \qquad |\mathbf{H}_{12}| \leq \mathbf{H}_{11}, \qquad \mathbf{H}_{11} > 0.$$

Furthermore, let $\bigcirc$ *be the general coordinate reference given in* (10) *then*

$$\begin{aligned}
&\mathbf{H}_{11}(\text{origin at } \bigcirc) \\
&= \frac{m(m-1)(2m-1)}{6} - m(p-1+r)(m-p-r) \\
&\quad - \frac{m}{m^2 - \left(\frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})}\right)^2} \left\{ \frac{m^2 (m+1-2(p+r))^2}{4} \right. \\
&\quad - (m+1-2(p+r)) \operatorname{Re}\left(\sum\nolimits_k (\text{origin at } \bigcirc) \overline{\sum (\text{origin at } \bigcirc)} \right) \\
&\quad \left. + \left| \sum\nolimits_k (\text{origin at } \bigcirc) \right|^2 \right\},
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
&\mathbf{H}_{12}(\text{origin at } \bigcirc) \\
&= \sum\nolimits_{k^2} (\text{origin at } \bigcirc) \\
&\quad - \frac{1}{m^2 - \left(\frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})}\right)^2} \left\{ m^2 (m+1-2(p+r)) \sum\nolimits_k (\text{origin at } \bigcirc) \right. \\
&\quad - \frac{m^2 (m+1-2(p+r))^2}{4} \sum (\text{origin at } \bigcirc) \\
&\quad \left. - \left(\sum\nolimits_k (\text{origin at } \bigcirc) \right)^2 \overline{\sum (\text{origin at } \bigcirc)} \right\}.
\end{aligned} \tag{30}$$

where

$$\sum (\text{origin at } \bigcirc) \triangleq \sum_{k=0}^{m-1} e^{i(k-(p-1+r))\Delta\omega} = \left(\frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})} \right) e^{i(\frac{m+1}{2}-(p+r))\Delta\omega}, \tag{31}$$

$$\sum\nolimits_k (\text{origin at } \bigcirc) \triangleq \sum_{k=0}^{m-1} (k-(p-1+r)) e^{i(k-(p-1+r)\Delta\omega} = -i \frac{d}{d\Delta\omega} \left(\sum (\text{origin at } \bigcirc) \right), \tag{32}$$

$$\sum\nolimits_{k^2} (\text{origin at } \bigcirc) \triangleq \sum_{k=0}^{m-1} (k-(p-1+r))^2 e^{i(k-(p-1+r))\Delta\omega} = -i \frac{d}{d\Delta\omega} \left(\sum\nolimits_k (\text{origin at } \bigcirc) \right). \tag{33}$$

For ease of notation we define

$$a \triangleq \frac{m}{m^2 - \left(\frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})}\right)^2}, \qquad b \triangleq \frac{\frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})}}{m^2 - \left(\frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})}\right)^2} \tag{34}$$

where

$$\wp(\Delta\omega; m) \triangleq \frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})}. \tag{35}$$

Notice that

$$m^2 - \left(\frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})}\right)^2 = \det(\mathbf{A}^*\mathbf{A}) = \det(\mathbf{A}_0^*\mathbf{A}_0) \; > \; 0 \tag{36}$$

since $\mathbf{A}_0^*\mathbf{A}_0$ is Hermitian (hence its determinant is non negative) and, for $0 < \Delta\omega \leq \pi$ and $m \geq 3$, the two columns of $\mathbf{A}_0$ are linearly independent (hence $\det(\mathbf{A}_0^*\mathbf{A}_0) \neq 0$). Thus,

$$\left|\frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})}\right| < m.$$

Therefore,

$$a > |b| \geq 0, \qquad \text{sign}(b) = \text{sign}(\wp(\Delta\omega; m)). \tag{37}$$

As a consequence of Lemma 3, if the coordinate origin is at the center of the array (hence $p + r = \frac{m+1}{2}$ regardless of the parity of m), it follows from (31) to (33) that $\mathbf{H}$ is a real matrix with entries

$$\mathbf{H}_{11}(\text{origin at center}) = \frac{m(m-1)(m+1)}{12} - a \cdot \left(\frac{d\wp}{d\Delta\omega}\right)^2, \tag{38}$$

and

$$\mathbf{H}_{12}(\text{origin at center}) = -\frac{d^2\wp}{d(\Delta\omega)^2} - b \cdot \left(\frac{d\wp}{d\Delta\omega}\right)^2. \tag{39}$$

Evidenced by (2), (4), (7), and (8), as well as in (9), the choice of the center of the array as the coordinate reference will greatly facilitate the investigation of the effects of the temporal phase difference on **CRB**, $\mathbf{C}_{MU}$, and $\mathbf{C}_{WSF}$ since the cross terms that involve both the temporal phase difference θ and $\arg(\mathbf{H}_{12})$ (dependent on the electrical angular separation $\Delta\omega$) will be simplified when $\mathbf{H}$ is real. Clearly, the analysis can be carried out in terms of the general coordinate reference (10). However, to avoid cumbersome arguments and expressions involving awkward translations, we elect to present results in the remainder of the paper for the coordinate origin chosen at the center of the array.

4. EFFECT OF TEMPORAL PHASE DIFFERENCE ON CRB

The main result in this section can be summarized as follows. Recall that, by Lemma 2, we can restrict the temporal phase difference θ to the interval $[0, \pi]$.

Proposition A *Consider* $[\mathbf{CRB}]_{ii}$, $i = 1,2$, *as a function of the temporal phase difference* θ, $0 \leq \theta \leq \pi$, *of two narrowband signals impinging on a ULA whose coordinate origin is chosen at the center of the array. Then* $[\mathbf{CRB}(\theta)]_{ii}$, $i = 1,2$, *is symmetric about* $\pi/2$. *Furthermore, it is decreasing for* $0 < \theta < \pi/2$, *increasing for* $\pi/2 < \theta < \pi$, *attains the maximum (either finite or infinite) at* 0 *and* π, *and the minimum at* $\pi/2$.

Detailed statements and proofs leading to this result are presented below in a sequence of two lemmas. Proposition A can be deduced from reference [15]; however, our argument is based on expression (2) derived in reference [2].

First, using the coordinate origin at the center of the array, we obtain from Lemma 3

$$\mathrm{Re}(\mathbf{H} \odot \mathbf{S}^T) = \sqrt{\pi_1 \pi_2} \begin{bmatrix} \sqrt{\frac{\pi_1}{\pi_2}}\, \mathbf{H}_{11} & |\rho|\, \mathbf{H}_{12}\, \mathrm{Re}(e^{-i\theta}) \\ |\rho|\, \mathbf{H}_{12}\, \mathrm{Re}(e^{i\theta}) & \sqrt{\frac{\pi_2}{\pi_1}}\, \mathbf{H}_{11} \end{bmatrix}$$

and then

$$\det\left(\mathrm{Re}(\mathbf{H} \odot \mathbf{S}^T)\right) = \pi_1 \pi_2 \left(\mathbf{H}_{11}{}^2 - |\rho|^2\, \mathbf{H}_{12}{}^2 \cos^2(\theta)\right). \tag{40}$$

We shall consider $[\mathbf{CRB}]_{ii}$, $i = 1,2$, as a function of θ. In view of Lemma 2, it suffices to restrict θ to the interval $0 \leq \theta \leq \pi$.

Using (2) and (40), we have

$$[\mathbf{CRB}]_{ii} \;=\; \frac{\sigma^2}{2\pi_i\, \mathbf{H}_{11}} \cdot \left(\frac{1}{1 - |\rho|^2 \left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2 \cos^2(\theta)} \right), \qquad i = 1,2. \tag{41}$$

Clearly,

Lemma 4 *Consider* $[\mathbf{CRB}]_{ii}$, $i = 1,2$, *of two narrowband plane waves, with electrical angular separation* $\Delta\omega$, *impinging on a ULA whose coordinate origin is chosen at the center of the array. Suppose that either the two signals are incoherent* ($\rho = 0$) *or* $\mathbf{H}_{12} = 0$, *then*

$$[\mathbf{CRB}]_{ii} \;=\; \frac{\sigma^2}{2\pi_i \mathbf{H}_{11}}, \qquad i = 1,2, \tag{42}$$

that is the variances $[\mathbf{CRB}]_{11}$ and $[\mathbf{CRB}]_{22}$ *are independent of* θ.

We note that, using (2) and (26), Lemma 4 holds for arbitrary coordinate reference where $\mathbf{H}_{11}$, $\mathbf{H}_{12}$ and $[\mathbf{CRB}]_{ii}$, $i = 1,2$, are measured with respect to that coordinate reference. Also, using (30) to (33), we observe that for any choice of coordinate reference and for any even $m \geq 4$, then $\mathbf{H}_{12} = 0$ when $\Delta\omega = \frac{m}{2}$ BW $(= \pi)$ where BW $\triangleq \frac{2\pi}{m}$ denotes a beamwidth.

Next, we have

Lemma 5 *Consider* $[\mathbf{CRB}]_{ii}$, $i = 1,2$, *as a function of* θ, *for* $0 \leq \theta \leq \pi$. *Then* $[\mathbf{CRB}(\theta)]_{ii}$, $i = 1,2$, *is symmetric about* $\frac{\pi}{2}$. *Moreover,* $[\mathbf{CRB}(\theta)]_{ii}$, $i = 1,2$, *is decreasing for* $0 < \theta < \pi/2$, *increasing for* $\pi/2 < \theta < \pi$, *attains the maximum at* $\theta = 0, \pi$, *and the minimum at* $\theta = \pi/2$, *with*

$$\min_{\theta} [\mathbf{CRB}(\theta)]_{ii} = \frac{\sigma^2}{2\pi_i \mathbf{H}_{11}}, \qquad i = 1,2, \tag{43}$$

$$\max_{\theta} [\mathbf{CRB}(\theta)]_{ii} = \frac{\sigma^2}{2\pi_i \mathbf{H}_{11}} \cdot \left(\frac{1}{1 - |\rho|^2 \left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2} \right), \qquad i = 1,2. \tag{44}$$

Furthermore, $[\mathbf{CRB}]_{ii}$, $i = 1,2$, *is arbitrarily large* (∞) *if and only if*

(i) $|\mathbf{H}_{12}| = \mathbf{H}_{11}$ *(hence,* $\mathbf{H}$ *is singular),*

(ii) $|\rho| = 1$ *(hence,* $\mathbf{S}$ *is singular i.e., the two signals are coherent), and*

(iii) $\theta = 0, \pi$ *(i.e., the two signals are in phase or out of phase).*

Proof. In view of Lemma 4, without loss of generality, we can assume $0 < |\rho| \leq 1$ and $\mathbf{H}_{12} \neq 0$. We notice that for any $\xi \in [0, \pi/2]$,

$$\cos^2\left(\frac{\pi}{2} \pm \xi)\right) = \sin^2(\xi).$$

Hence, for $0 \leq \theta \leq \pi$, the variance $[\mathbf{CRB}(\theta)]_{ii}$, $i = 1,2$, is symmetric about $\frac{\pi}{2}$. The rest of the lemma follows from examining the factor

$$\frac{1}{1 - |\rho|^2 \left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2 \cos^2(\theta)}$$

of $[\mathbf{CRB}]_{ii}$, $i = 1,2$, in (41) as a function of θ. ■

Qualitatively, when the normalized correlation magnitude $|\rho|$ is large (close to unity) then it follows from (41) that, other things being equal, effect of θ on $[\mathbf{CRB}]_{ii}$, $i = 1,2$, is intensified.

On the other hand, when the electrical angular separation $\Delta\omega$ is small in terms of beamwidth, (equivalently, the two steering vectors which are columns of $\mathbf{A}$ are close), then the two columns of $\mathbf{D}$ (being elementwise derivatives of the two steering vectors) are close. Since

$$\mathbf{H} \triangleq \mathbf{D}^* \mathbf{P}_{\mathbf{A}}^{\perp} \mathbf{D} = \mathbf{D}^* \left(\mathbf{P}_{\mathbf{A}}^{\perp}\right)^* \mathbf{P}_{\mathbf{A}}^{\perp} \mathbf{D} = \left(\mathbf{P}_{\mathbf{A}}^{\perp} \mathbf{D}\right)^* \mathbf{P}_{\mathbf{A}}^{\perp} \mathbf{D},$$

it follows that the ratio $|\mathbf{H}_{12}|/\mathbf{H}_{11}$ is close to unity when $\Delta\omega$ is small. Also, from (38) we notice that $\mathbf{H}_{11}$ is small when $\Delta\omega$ is small. Consequently, other things being equal, from (41) we see immediately that effect of θ on $[\mathbf{CRB}]_{ii}$, $i = 1,2$, is augmented when $\Delta\omega$ is small.

Therefore, the effect of temporal phase difference on $[\mathbf{CRB}]_{ii}$, $i = 1,2$, are intensified when the two signals are closely spaced and/or their normalized correlation magnitude is high.

For the general coordinate reference, we notice that $\mathbf{H}$ need not be real and thus (41) is replaced by

$$[\mathbf{CRB}]_{ii} = \frac{\sigma^2}{2\pi_i \mathbf{H}_{11}} \cdot \left(\frac{1}{1 - |\rho|^2 \left(\frac{|\mathbf{H}_{12}|}{\mathbf{H}_{11}}\right)^2 \cos^2(\arg(\mathbf{H}_{12}) - \theta)} \right), \quad i = 1,2. \tag{45}$$

In Appendix B, we present an example where the electrical angular separation $\Delta\omega$ is an integer multiple of (standard or first null) beamwidth, denoted by BW, with one BW equal to $2\pi/m$. That is

$$\Delta\omega = \ell\,\mathrm{BW} = \ell \left(\frac{2\pi}{m}\right),$$

where $\ell = 1,2,\ldots,\lfloor\frac{m}{2}\rfloor$ with $\lfloor\frac{m}{2}\rfloor$ denotes the largest integer less than or equal to $\frac{m}{2}$. It is interesting to note that, in this situation, $\mathbf{H}$ is nonsingular for $m \geq 4$ and singular for $m = 3$. Therefore, for $m \geq 4$ and $\Delta\omega = \ell$ BW, $\ell = 1,2,\ldots,\lfloor\frac{m}{2}\rfloor$, the variance $[\mathbf{CRB}]_{ii}$, $i = 1,2$, is finite for arbitrary signal correlation, whereas, when $m = 3$ and $\Delta\omega = 1\,\mathrm{BW}$, the variance $[\mathbf{CRB}]_{ii}$, $i = 1,2$, is arbitrarily large when the two signals are correlated with an appropriate temporal phase difference θ depending on the coordinate reference.

5. EFFECT OF TEMPORAL PHASE DIFFERENCE ON C_{MU}

With the coordinate reference at the center of the array, by the computation in Appendix C we have

$$[\mathbf{C}_{MU}]_{ii} = \frac{\sigma^2}{2\,\pi_i\,\mathbf{H}_{11}\,(1-|\rho|^2)} \cdot (A_i + B_i\,\cos(\theta)) \tag{46}$$

where

$$A_i \;=\; 1 \;+\; \left(\pi_j + |\rho|^2\,\pi_i\right)\cdot a, \tag{47}$$

$$B_i \;=\; 2\sqrt{\pi_1\pi_2}\,|\rho|\cdot b, \tag{48}$$

with

$$\mathbf{H}_{11} \quad \text{given in (38),}$$

$$a \text{ and } b \quad \text{given in (34),}$$

$$\det(\mathbf{S}) = \pi_1\pi_2\left(1-|\rho|^2\right) > 0 \qquad \text{and} \qquad i,j\in\{1,2\},\; j\neq i.$$

Notice that A_i and B_i, $i = 1,2,$ are independent of the temporal phase difference θ. Also, using the fact that $2\sqrt{\pi_1\pi_2}\,|\rho| \;\leq\; \pi_j \;+\; |\rho|^2\,\pi_i,\;\; i,j = 1,2,\, j\neq i$ (as "geometric mean" $\leq$ "arithmetic mean" or, simply, by a quadratic expansion) in conjunction with (37), we obtain, for $i = 1,2,$

$$A_i \;>\; |B_i| \;\geq\; 0, \qquad \operatorname{sign}(B_i) \;=\; \operatorname{sign}(\wp(\Delta\omega;m)).$$

If $\rho = 0$ then $B_i = 0$ and, using Lemma 4, we obtain immediately

Proposition B.1 *Consider* $[\mathbf{C}_{MU}]_{ii}$, $i = 1,2,$ *of two narrowband plane waves, with electrical angular separation* $\Delta\omega$, *impinging on a ULA whose coordinate origin is chosen at the center of the array. Suppose that the two signals are incoherent* ($\rho = 0$) *then, for* $i = 1,2,$

$$\begin{aligned}[\mathbf{C}_{MU}]_{ii} \;&=\; \frac{\sigma^2}{2\,\pi_i\,\mathbf{H}_{11}}\cdot\left(1 \;+\; a\cdot\frac{\sigma^2}{\pi_i}\right)\\ &=\; \min_{\theta}\,[\mathbf{CRB}(\theta)]_{ii} \;+\; 2\,\mathbf{H}_{11}\cdot a\cdot\left(\min_{\theta}\,[\mathbf{CRB}(\theta)]_{ii}\right)^2.\end{aligned}$$

Next, we consider the general case $(0 \leq |\rho| < 1)$ where Proposition B.1 depicts a degenerate situation. For any $0 < \Delta\omega \leq \pi$, there exists a nonnegative integer ℓ, with $0 \leq \ell \leq \lfloor \frac{m}{2} \rfloor$, such that $\ell\,\mathrm{BW} < \Delta\omega \leq \min\{(\ell+1)\,\mathrm{BW}, \pi\}$; where $\mathrm{BW} \stackrel{\Delta}{=} 2\pi/m$ denotes a beamwidth, and $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Clearly, for $i = 1, 2$,

$$
\begin{aligned}
B_i &> 0, \qquad \text{for } \ell\,\mathrm{BW} < \Delta\omega < (\ell+1)\,\mathrm{BW} \text{ and } \ell \text{ even (including zero)}, \\
B_i &< 0, \qquad \text{for } \ell\,\mathrm{BW} < \Delta\omega < (\ell+1)\,\mathrm{BW} \text{ and } \ell \text{ odd}, \\
B_i &= 0, \qquad \text{for } \Delta\omega = (\ell+1)\,\mathrm{BW}.
\end{aligned}
$$

Therefore,

Proposition B.2 *Consider $[\mathbf{C}_{MU}]_{ii}$, $i = 1, 2$, as a function of the temporal phase difference θ, $0 \leq \theta \leq 2\pi$, of two narrowband, noncoherent $(0 \leq |\rho| < 1)$ plane waves, with electrical angular separation $\Delta\omega$, impinging on a ULA whose coordinate origin is chosen at the center of the array. Suppose $\ell\,\mathrm{BW} < \Delta\omega \leq \min\{(\ell+1)\,\mathrm{BW}, \pi\}$ for some nonnegative integer ℓ with $0 \leq \ell \leq \lfloor \frac{m}{2} \rfloor$. Then $[\mathbf{C}_{MU}(\theta)]_{ii}$, $i = 1, 2$, is symmetric about $\theta = \pi$. Moreover,*

(a) *for $\ell\,\mathrm{BW} < \Delta\omega < \min\{(\ell+1)\,\mathrm{BW}, \pi\}$,*

- *if ℓ is even (including zero), then $[\mathbf{C}_{MU}(\theta)]_{ii}$, $i = 1, 2$, is decreasing for $0 < \theta < \pi$, increasing for $\pi < \theta < 2\pi$, attains its maximum at $\theta = 0,\ 2\pi$, and its minimum at $\theta = \pi$;*
- *if ℓ is odd, then $[\mathbf{C}_{MU}(\theta)]_{ii}$, $i = 1, 2$, is increasing for $0 < \theta < \pi$, decreasing for $\pi < \theta < 2\pi$, attains its minimum at $\theta = 0,\ 2\pi$, and its maximum at $\theta = \pi$;*

in either case,

$$
\begin{aligned}
\min_\theta\,[\mathbf{C}_{MU}(\theta)]_{ii} &= \frac{\sigma^2}{2\,\pi_i\,\mathbf{H}_{11}\,(1 - |\rho|^2)} \cdot (A_i - |B_i|), \qquad i = 1, 2 \\
\max_\theta\,[\mathbf{C}_{MU}(\theta)]_{ii} &= \frac{\sigma^2}{2\,\pi_i\,\mathbf{H}_{11}\,(1 - |\rho|^2)} \cdot (A_i + |B_i|), \qquad i = 1, 2
\end{aligned}
$$

where A_i and B_i are given in (47) *and* (48);

(b) *for $\Delta\omega = (\ell+1)\,\mathrm{BW}\ (\leq \pi)$,*

$$
[\mathbf{C}_{MU}(\theta)]_{ii} = \frac{\sigma^2}{2\,\pi_i\,(1 - |\rho|^2)\ \mathbf{H}_{11}|_{\Delta\omega=(\ell+1)\,\mathrm{BW}}} \cdot A_i|_{\Delta\omega=(\ell+1)\,\mathrm{BW}}\,, \qquad i = 1, 2,
$$

where

$$A_i|_{\Delta\omega=(\ell+1)\,\mathrm{BW}} \;=\; 1 + \frac{\sigma^2}{m\,\det(\mathbf{S})}\cdot\left(\pi_j + |\rho|^2\,\pi_i\right), \qquad i,j=1,2 \text{ and } j\neq i,$$

and

$$\mathbf{H}_{11}|_{\Delta\omega=(\ell+1)\,\mathrm{BW}} \;=\; \frac{m}{4}\left[\frac{m^2-1}{3} - \frac{1}{\sin^2\left(\frac{(\ell+1)\pi}{m}\right)}\right];$$

hence, it is independent of θ.

Furthermore, in all cases, when $|\rho|$ approaches unity, $[\mathbf{C}_{MU}(\theta)]_{ii}$ gets arbitrarily large, regardless of the temporal phase difference θ and electrical angular separation $\Delta\omega$ or any other parameters.

It follows immediately from (46) to (48) that for large $|\rho|$, other things being equal, effect of θ on $[\mathbf{C}_{MU}]_{ii}$, $i=1,2$, is intensified.

On the other hand, when $\Delta\omega$ is small (in terms of beamwidth), $\mathbf{H}_{11}$ is small and, by (34), a and b are large. Therefore, effect of θ on $[\mathbf{C}_{MU}]_{ii}$, $i=1,2$, is intensified when $\Delta\omega$ is small.

Consequently, effect of the temporal phase difference on $[\mathbf{C}_{MU}]_{ii}$, $i=1,2$, is intensified when the two signals are closely spaced and/or their normalized correlation magnitude is high.

6. EFFECT OF TEMPORAL PHASE DIFFERENCE ON C_{ML}

From (9), with the coordinate reference at the center of the array, a straightforward, but tedious, computation yields

$$
\begin{aligned}
[\mathbf{C}_{ML}]_{ii} = \quad & [\mathbf{CRB}]_{ii} + 2\,\mathbf{H}_{11}\cdot a\cdot\left([\mathbf{CRB}]_{ii}{}^{2}+[\mathbf{CRB}]_{12}{}^{2}\right)\\
& -4\,\mathbf{H}_{12}\cdot b\cdot[\mathbf{CRB}]_{ii}\cdot[\mathbf{CRB}]_{12}, \qquad i=1,2,
\end{aligned}
\tag{49}
$$

where

$$
[\mathbf{CRB}]_{ii} = \frac{\sigma^2}{2\pi_i\,\mathbf{H}_{11}}\cdot\left(\frac{1}{1-|\rho|^2\left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2\cos^2(\theta)}\right), \qquad i=1,2, \tag{50}
$$

$$
[\mathbf{CRB}]_{12} = \frac{-\sigma^2\,|\rho|\,\mathbf{H}_{12}}{2\sqrt{\pi_1\pi_2}\,\mathbf{H}_{11}{}^2}\cdot\left(\frac{\cos(\theta)}{1-|\rho|^2\left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2\cos^2(\theta)}\right). \tag{51}
$$

Clearly, if either $\rho = 0$ or $\mathbf{H}_{12} = 0$, then $[\mathbf{CRB}(\theta)]_{12} = 0$ and $[\mathbf{CRB}(\theta)]_{ii} = \min_\theta[\mathbf{CRB}(\theta)]_{ii}$, $i = 1,2$. Therefore,

Proposition C.1 *Consider* $[\mathbf{C}_{ML}]_{ii}$, $i = 1,2$, *of two narrowband plane waves, with electrical angular separation* $\Delta\omega$, *impinging on a ULA whose coordinate origin is chosen at the center of the array. Suppose that either the two signals are incoherent* ($\rho = 0$) *or* $\mathbf{H}_{12} = 0$ *then, for* $i = 1,2$,

$$
[\mathbf{C}_{ML}]_{ii} = \min_\theta[\mathbf{CRB}(\theta)]_{ii} + 2\,\mathbf{H}_{11}\cdot a\cdot\left(\min_\theta[\mathbf{CRB}(\theta)]_{ii}\right)^2,
$$

which is independent of θ *and equal to* $[\mathbf{C}_{MU}]_{ii}$ when $\rho = 0$.

Proposition C.1 is consistent with the general result of Stoica and Nehorai (reference [2]), which states that if the matrix S (not necessarily a 2×2 matrix) is diagonal then the asymptotic covariance of ML coincides with that of MUSIC.

We next consider the general case ($0 \le |\rho| \le 1$ and $0 \le |\mathbf{H}_{12}| \le \mathbf{H}_{11}$) where Proposition C.1. is a degenerate situation.

First, we suppose that $0 \le |\rho| < 1$. To facilitate the discussion, we rewrite the expression (49) as follows.

$$
[\mathbf{C}_{ML}(\theta)]_{ii} = T_1(\theta) + T_2(\theta) \tag{52}
$$

where

$$T_1(\theta) \;=\; [\mathbf{CRB}(\theta)]_{ii} \;+\; 2\,\mathbf{H}_{11}\cdot a\cdot\left([\mathbf{CRB}(\theta)]_{ii}{}^2+[\mathbf{CRB}(\theta)]_{12}{}^2\right), \tag{53}$$

$$T_2(\theta) \;=\; \frac{\sigma^4\,|\rho|\,\mathbf{H}_{12}{}^2}{\pi_i\sqrt{\pi_1\pi_2}\,\mathbf{H}_{11}{}^3}\cdot b\cdot\frac{\cos(\theta)}{\left(1-|\rho|^2\left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2\cos^2(\theta)\right)^2}. \tag{54}$$

Notice that

$$\frac{dT_1}{d\theta} \;=\; \frac{\sigma^2}{\pi_i\,\mathbf{H}_{11}}\cdot\frac{-\sin(\theta)}{\left(1-|\rho|^2\left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2\cos^2(\theta)\right)^2}\times F(\theta)\cos(\theta), \tag{55}$$

$$\frac{dT_2}{d\theta} \;=\; \frac{\sigma^2}{\pi_i\,\mathbf{H}_{11}}\cdot\frac{-\sin(\theta)}{\left(1-|\rho|^2\left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2\cos^2(\theta)\right)^2}\times G(\theta), \tag{56}$$

where

$$\begin{aligned} F(\theta) \;=\; & |\rho|^2\left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2\left\{1\;+\;a\cdot\left[\frac{\sigma^2}{\pi_i}\left(\frac{2}{1-|\rho|^2\left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2\cos^2(\theta)}\right)\right.\right. \\ & \left.\left.+\;\frac{\sigma^2}{\pi_j}\left(\frac{1+|\rho|^2\left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2\cos^2(\theta)}{1-|\rho|^2\left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2\cos^2(\theta)}\right)\right]\right\}, \qquad i,j=1,2 \text{ and } j\neq i, \end{aligned}$$

and

$$G(\theta) \;=\; \frac{\sigma^2\,|\rho|\left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2}{\sqrt{\pi_1\pi_2}}\cdot b\cdot\left(1\;+\;\frac{4\,|\rho|^2\left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2\cos^2(\theta)}{1\;-\;|\rho|^2\left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2\cos^2(\theta)}\right).$$

Obviously, if either $\rho = 0$ or $\mathbf{H}_{12} = 0$ then both $F(\theta)$ and $G(\theta)$ vanish, and hence $[\mathbf{C}_{ML}(\theta)]_{ii}$ is independent of θ as known in Proposition C.1.

For $\rho \neq 0$ and $\mathbf{H}_{12} \neq 0$ we have immediately

$$F(\theta) \;>\; 0, \qquad \operatorname{sign}(G(\theta)) \;=\; \operatorname{sign}(\wp(\Delta\omega;m)).$$

Thus, $T_1(\theta)$ is positive and, as a function of θ, behaves exactly like $[\mathbf{CRB}(\theta)]_{ii}$. The term $T_2(\theta)$ of (52) is more troublesome due to the factor $\wp(\Delta\omega;m)$.

Recall that, for any $0 < \Delta\omega \leq \pi$, there exists a nonnegative integer ℓ, with $0 \leq \ell \leq \lfloor\frac{m}{2}\rfloor$, such that $\ell\,\mathrm{BW} < \Delta\omega \leq \min\{(\ell+1)\,\mathrm{BW},\pi\}$; where $\mathrm{BW} \stackrel{\Delta}{=} 2\pi/m$ denotes a beamwidth and $\lfloor x\rfloor$ denotes the largest integer less than or equal to x. Clearly,

$$\wp(\Delta\omega;m)\quad\begin{cases} >\;0, & \text{for } \ell\,\mathrm{BW} < \Delta\omega < (\ell+1)\,\mathrm{BW} \text{ and } \ell \text{ even (including zero)},\\ <\;0, & \text{for } \ell\,\mathrm{BW} < \Delta\omega < (\ell+1)\,\mathrm{BW} \text{ and } \ell \text{ odd},\\ =\;0, & \text{for } \Delta\omega = (\ell+1)\,\mathrm{BW}.\end{cases}$$

Suppose that $\wp(\Delta\omega; m) > 0$. Clearly, the global maximum occurs at $\theta = 0,\ 2\pi$. From (55) and (56) we have

$$\frac{dT_1}{d\theta} \quad \begin{cases} < 0, & 0 < \theta < \pi/2, \text{ and } \pi < \theta < 3\pi/2, \\ = 0, & \theta = \pi/2,\ \pi,\ 3\pi/2,\ 2\pi, \\ > 0, & \pi/2 < \theta < \pi, \text{ and } 3\pi/2 < \theta < 2\pi, \end{cases}$$

$$\frac{dT_2}{d\theta} \quad \begin{cases} < 0, & 0 < \theta < \pi, \\ = 0, & \theta = 0,\ \pi,\ 2\pi, \\ > 0, & \pi < \theta < 2\pi, \end{cases}$$

thus

$$\frac{d}{d\theta}[\mathbf{C}_{ML}(\theta)]_{ii} \;=\; \frac{dT_1}{d\theta} + \frac{dT_2}{d\theta} \quad \begin{cases} < 0, & 0 < \theta \leq \pi/2, \\ > 0, & 3\pi/2 \leq \theta < 2\pi, \end{cases}$$

which implies $[\mathbf{C}_{ML}(\theta)]_{ii}$ is decreasing for $0 < \theta \leq \pi/2$ and increasing for $3\pi/2 \leq \theta < 2\pi$.

Now consider $\pi/2 < \theta < 3\pi/2$. By the symmetry of $[\mathbf{C}_{ML}(\theta)]_{ii}$ about π we need only examine the case where $\pi/2 < \theta \leq \pi$.

We write

$$\frac{d}{d\theta}\left([\mathbf{C}_{ML}(\theta)]_{ii}\right) \;=\; \frac{\sigma^2}{\pi_i \mathbf{H}_{11}} \cdot \frac{-\sin(\theta)}{\left(1 - |\rho|^2 \left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2 \cos^2(\theta)\right)^2} \times \left(F(\theta)\cos(\theta) + G(\theta)\right).$$

Suppose that $\frac{d}{d\theta}\left([\mathbf{C}_{ML}(\theta)]_{ii}\right)$ is not identically zero on $(\pi/2, \pi]$, that is $\rho \neq 0$ and $\mathbf{H}_{12} \neq 0$. If $\frac{d}{d\theta}\left([\mathbf{C}_{ML}(\theta)]_{ii}\right)$ vanishes at any point $\theta_0 \in (\pi/2, \pi)$, then θ_0 is a zero of the function $F(\theta)\cos(\theta) + G(\theta)$ and

$$\frac{|G(\theta)|}{F(\theta)} \;=\; \frac{1}{|\rho|} \times \frac{\text{num}}{\text{denom},} \;\leq\; 1$$

where

$$\text{num} \;\triangleq\; \sigma^2 \sqrt{\pi_1 \pi_2} \left(\frac{|b|}{m^2 - \left(\frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})}\right)^2} \right) \left(1 + 3|\rho|^2 \left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2 \cos^2(\theta)\right)$$

and

$$\begin{aligned} \text{denom} \;\triangleq\; & \pi_1 \pi_2 \left(1 - |\rho|^2 \left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2 \cos^2(\theta)\right) \\ & + \sigma^2 \cdot a \cdot \left(2\pi_j + \pi_i \left(1 + |\rho|^2 \left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2 \cos^2(\theta)\right)\right). \end{aligned}$$

Notice that for either small $|\rho|$, or small $|\mathbf{H}_{12}|/\mathbf{H}_{11}$ or for θ being close to $\pi/2$, then

$$\frac{\textbf{num}}{\textbf{denom}} \approx \gamma \triangleq \frac{\sigma^2 \sqrt{\pi_1\pi_2} \cdot |b|}{\pi_1\pi_2 + \sigma^2 \cdot a \cdot (2\pi_j + \pi_i)} < \frac{1}{2} \tag{57}$$

since

$$\left|\frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})}\right| < m, \qquad \text{and} \qquad \sqrt{\pi_1\pi_2} \leq \frac{\pi_1+\pi_2}{2}.$$

Therefore, if $\frac{\gamma}{|\rho|}$ is close to zero then $\frac{d}{d\theta}([\mathbf{C}_{ML}(\theta)]_{ii})$ vanishes at some $\theta_0 \in (\pi/2, \pi)$ which is close to $\pi/2$. The case where the normalized correlation magnitude $|\rho|$ is large and the SNR's $\frac{\pi_1}{\sigma^2}$ and $\frac{\pi_2}{\sigma^2}$ are high belongs to this situation.

On the other hand, $0 < |\rho| \ll \gamma$ then $\frac{d}{d\theta}([\mathbf{C}_{ML}(\theta)]_{ii})$ does not vanish on $(\pi/2, \pi)$. Since

$$\frac{d}{d\theta}([\mathbf{C}_{ML}(\theta)]_{ii})\Big|_{\theta=\pi/2} < 0, \qquad \frac{d}{d\theta}([\mathbf{C}_{ML}(\theta)]_{ii})\Big|_{\theta=\pi} = 0$$

and since $\frac{d}{d\theta}([\mathbf{C}_{ML}(\theta)]_{ii})$ is a continuous function in θ, it follows that $\frac{d}{d\theta}([\mathbf{C}_{ML}(\theta)]_{ii}) < 0$ on $(\pi/2, \pi)$. Consequently, $[\mathbf{C}_{ML}(\theta)]_{ii}$ is decreasing on $(\pi/2, \pi)$ and, by its symmetry about π, increasing on $(\pi, 3\pi/2)$ with the minimum achieved at $\theta = \pi$.

Thus, the behavior of $[\mathbf{C}_{ML}(\theta)]_{ii}$ on $(\pi/2, 3\pi/2)$ is quite intricate due to the interaction of different parameters. In general, since

$$\frac{d}{d\theta}([\mathbf{C}_{ML}(\theta)]_{ii}) < 0, \qquad \theta \in (0, \pi/2],$$

$$\frac{d}{d\theta}([\mathbf{C}_{ML}(\theta)]_{ii})\Big|_{\theta=\pi} = 0,$$

and

$$\frac{d}{d\theta}([\mathbf{C}_{ML}(\theta)]_{ii}) > 0, \qquad \theta \in [3\pi/2, 2\pi),$$

and since $([\mathbf{C}_{ML}(\theta)]_{ii})$ attains its (global) maximum at $\theta = 0, 2\pi$, it follows that there is a (global) minimum at $\theta_0 \in (\pi/2, \pi]$ and, by its symmetry about π, also at $2\pi - \theta_0$. The exact location of θ_0 however depends of $\Delta\omega$ and other parameters π_1/σ^2, π_2/σ^2, $|\rho|$, and m.

A similar argument can also be applied to the case where $\wp(\Delta\omega; m) < 0$.

Finally, we note from (49) (also from (9)) that, for $0 \leq |\rho| < 1$, the variance $[\mathbf{C}_{ML}]_{ii}$, $i = 1, 2$, can be expressed in terms of elements of the matrix **CRB** which hold for $0 \leq |\rho| \leq 1$. Therefore, the following result, which is a summary of the preceding discussion, also holds when $|\rho| = 1$ by an elementary continuity argument.

Proposition C.2 *Consider* $[\mathbf{C}_{ML}]_{ii}$, $i = 1, 2$, *as a function of the temporal phase difference* θ, $0 \leq \theta \leq 2\pi$, *of two narrowband plane waves with arbitrary signal corelation*

$(0 \leq |\rho| \leq 1)$ *and electrical angular separation* $\Delta\omega$, *impinging on a ULA whose coordinate origin is chosen at the center of the array. Suppose* $\ell\,\mathrm{BW} < \Delta\omega \leq \min\{(\ell+1)\,\mathrm{BW}, \pi\}$ *for some nonnegative integer* ℓ *with* $0 \leq \ell \leq \lfloor \frac{m}{2} \rfloor$. *Then*

(a) *for* $\ell\,\mathrm{BW} < \Delta\omega < \min\{(\ell+1)\,\mathrm{BW}, \pi\}$, *then* $[\mathbf{C}_{ML}(\theta)]_{ii}$, $i = 1, 2$, *is symmetric about* $\theta = \pi$ *and*

- *if* ℓ *is even (including zero) then, on the interval* $[0, \pi]$, $[\mathbf{C}_{ML}(\theta)]_{ii}$ *attains its (global) maximum at* $\theta = 0$, *and its (global) minimum at* $\theta_0 \in (\pi/2, \pi]$;
- *if* ℓ *is odd then, on the interval* $[0, \pi]$, $[\mathbf{C}_{ML}(\theta)]_{ii}$, $i = 1, 2$, *attains its (global) maximum at* $\theta = \pi$, *and its (global) minimum at* $\theta_0 \in [0, \pi/2)$;

In either case, the exact location of θ_0 *depends on* $\Delta\omega$ *and other parameters* $\pi_1/\sigma^2, \pi_2/\sigma^2$, $|\rho|$, *and* m;

(b) *for* $\Delta\omega = (\ell+1)\,\mathrm{BW}\ (\leq \pi)$,

$$
\begin{aligned}
[\mathbf{C}_{ML}(\theta)]_{ii} \;=\; & \frac{\sigma^2}{2\,\pi_i\ \mathbf{H}_{11}|_{\Delta\omega=(\ell+1)\,\mathrm{BW}}} \cdot \Bigg[\left(1 - \frac{\sigma^2}{m\,\pi_j}\right) \cdot \frac{1}{1 - |\rho|^2 \left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2 \cos^2(\theta)} \\
& + \frac{\sigma^2\,(\pi_1 + \pi_2)}{m\pi_1\pi_2} \cdot \frac{1}{\left(1 - |\rho|^2 \left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2 \cos^2(\theta)\right)^2} \Bigg]_{\Delta\omega=(\ell+1)\,\mathrm{BW}}, \quad i, j = 1, 2;\ j \neq i,
\end{aligned}
$$

where

$$
\mathbf{H}_{11}|_{\Delta\omega=(\ell+1)\,\mathrm{BW}} = \frac{m}{4}\left[\frac{m^2-1}{3} - \frac{1}{\sin^2\left(\frac{(\ell+1)\pi}{m}\right)}\right],
$$

and

$$
\mathbf{H}_{12}|_{\Delta\omega=(\ell+1)\,\mathrm{BW}} = \frac{(-1)^{\ell+1}\, m \cos\left(\frac{(\ell+1)\pi}{m}\right)}{2\sin^2\left(\frac{(\ell+1)\pi}{m}\right)};
$$

thus, $[\mathbf{C}_{ML}(\theta)]_{ii}$ *is periodic of period* π *and behaves similarly to* $[\mathbf{CRB}(\theta)]_{ii}$, *that is, on the interval* $[0, \pi]$, *it is symmetric about* $\pi/2$, *decreasing for* $0 < \theta < \pi/2$, *increasing for* $\pi/2 < \theta < \pi$, *attains its maximum at* $\theta = 0,\ \pi$, *and its minimum at* $\theta = \pi/2$.

Furthermore, in all cases, $[\mathbf{C}_{ML}]_{ii}$, $i = 1, 2$, *is arbitrarily large* (∞) *if and only if*

(i) $|\mathbf{H}_{12}| = \mathbf{H}_{11}$ *(hence,* $\mathbf{H}$ *is singular),*

(ii) $|\rho| = 1$ *(hence,* $\mathbf{S}$ *is singular i.e., the two signals are coherent), and*

(iii) $\theta = 0,\ \pi$ *(i.e., the two signals are in phase or out of phase).*

Qualitatively, it follows from (49), using (50) and (51), that effect of the temporal phase difference on $[\mathbf{C}_{ML}]_{ii}$, $i = 1, 2$, is intensified when the normalized correlation magnitude $|\rho|$ is high.

On the other hand, since

$$\mathbf{H}_{11} \cdot [\mathbf{CRB}]^2_{ii} = \frac{\sigma^4}{4\pi_i^2\, \mathbf{H}_{11}} \left(\frac{1}{1 - |\rho|^2 \left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2 \cos^2(\theta)} \right)^2, \qquad i = 1, 2,$$

$$\mathbf{H}_{11} \cdot [\mathbf{CRB}]^2_{12} = \frac{\sigma^4\, |\rho|^2}{4\pi_1\pi_2\, \mathbf{H}_{11}} \cdot \left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2 \cdot \left(\frac{\cos(\theta)}{1 - |\rho|^2 \left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2 \cos^2(\theta)} \right)^2,$$

$$-\,4\,\mathbf{H}_{12} \cdot [\mathbf{CRB}]_{ii} \cdot [\mathbf{CRB}]_{12} = \frac{\sigma^4\, |\rho|}{\pi_i \sqrt{\pi_1\pi_2}\, \mathbf{H}_{11}} \cdot \left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2 \cdot \frac{\cos(\theta)}{\left(1 - |\rho|^2 \left(\frac{\mathbf{H}_{12}}{\mathbf{H}_{11}}\right)^2 \cos^2(\theta)\right)^2},$$

and since $\mathbf{H}_{11}$ is small and a, b are large when $\Delta\omega$ is small (in terms of beamwidth), it follows from (49) that, other things being equal, effect of θ on $[\mathbf{C}_{ML}]_{ii}$, $i = 1, 2$, is augmented when the two signals are closely spaced.

In short, effects of the temporal phase difference on $[\mathbf{C}_{ML}]_{ii}$, $i = 1, 2$, are intensified when the two signals are closely spaced and/or their normalized correlation magnitude is high.

7. NUMERICAL RESULTS

For simplicity, we shall consider the situation of two plane waves of equal powers impinging on a uniform linear array. By setting $\pi_1 = \pi_2$ in (41), (46) to (48) and (49) to (51), we have

$$[\mathbf{CRB}]_{11} = [\mathbf{CRB}]_{22}, \qquad [\mathbf{C}_{MU}]_{11} = [\mathbf{C}_{MU}]_{22}, \qquad [\mathbf{C}_{ML}]_{11} = [\mathbf{C}_{ML}]_{22}. \tag{58}$$

This, however, can be obtained without actually computing the corresponding variances as in Appendix A.

In figures 1(a) to 3(d), we consider (square-roots of) $[\mathbf{CRB}]_{ii}$, $[\mathbf{C}_{MU}]_{ii}$, and $[\mathbf{C}_{ML}]_{ii}$ as functions of two variables, the electrical angular separation $\Delta\omega$ and the temporal phase difference θ where

$$0.1\,\mathrm{BW} \leq \Delta\omega \leq 1\,\mathrm{BW}, \qquad \text{and} \qquad 0^0 \leq \theta \leq 360^0.$$

The vertical axis of each three-dimensional figure depicts $\sqrt{N}$ times of the corresponding standard deviation (see (3)), measured in BWs, where N, the number of time samples (snapshots), is assumed to be large.

Furthermore, all figures have the following common parameters

$$m = 5,$$

$$\frac{\pi_1}{\sigma^2} = \frac{\pi_2}{\sigma^2} = 20\,\mathrm{dB}.$$

The first set of figures (figures 1(a) to 1(d)) presents $\sqrt{[\mathbf{CRB}]_{ii}}$, for $|\rho| = 0.50$, 0.90, 0.95, and 1.00, respectively. The second set (figures 2(a) to 2(d)) presents $\sqrt{[\mathbf{C}_{MU}]_{ii}}$ for $|\rho| = 0.50$, 0.90, 0.95, and 0.97, respectively. Finally, $\sqrt{[\mathbf{C}_{ML}]_{ii}}$ is presented in the third set (figures 3(a) to 3(d)) for $|\rho| = 0.50$, 0.90, 0.95, and 1.00, respectively.

Figures 1(a) to 1(c), 2(a) to 2(d), and 3(a) to 3(c) are plotted using general formulas (2), (4), and (9), respectively; figure 1(d) is based on (2) and figure 3(d) on (6) together with (7) and (8). Thus, this graphical sample can be used to cross-examine the consistency of our results with respect to general formulas given in references [2], [3], and [6].

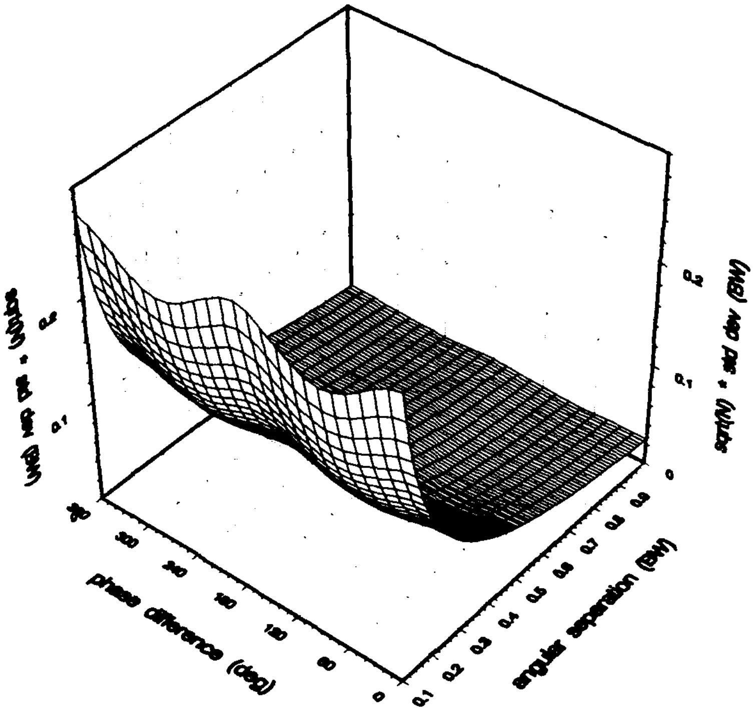

Figure 1(a). CRB for two equipowered signals
with |rho| =0.50, SNR=20dB, impinging on a 5-element ULA

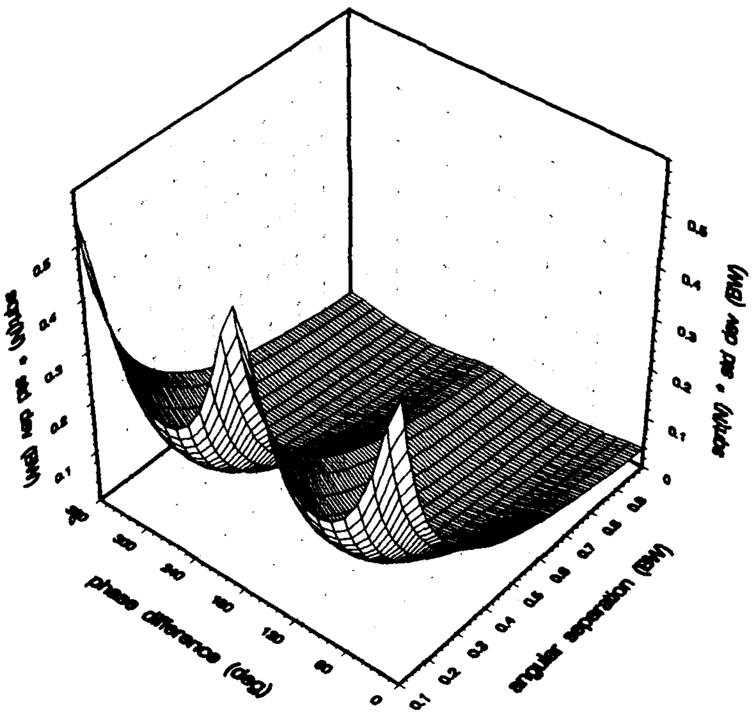

Figure 1(b). CRB for two equipowered signals
with |rho| =0.90, SNR=20 dB, impinging on a 5-element ULA

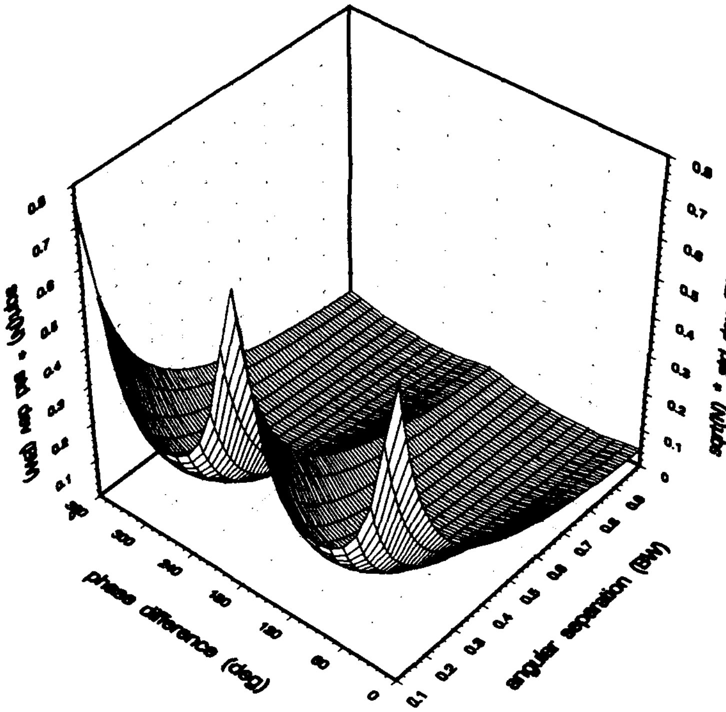

Figure 1(c). CRB for two equipowered signals
with |rho|=0.95, SNR=20 dB, impinging on a 5-element ULA

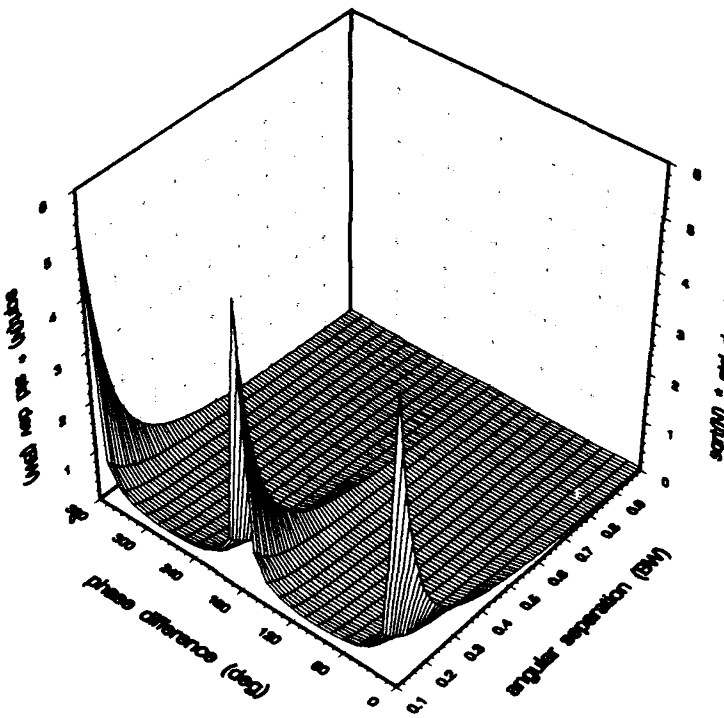

Figure 1(d). CRB for two equipowered signals
with |rho|=1.00, SNR=20 dB, impinging on a 5-element ULA

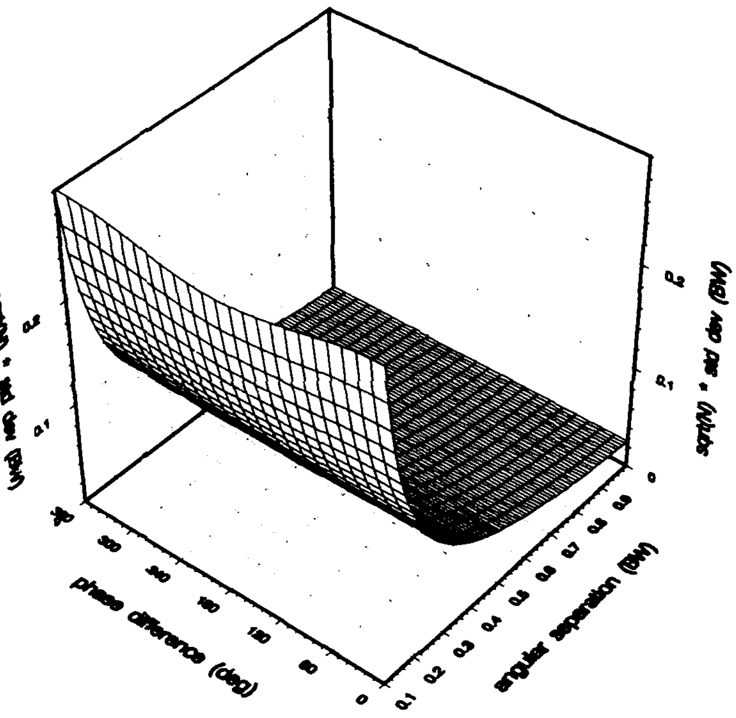

Figure 2(a). MUSIC for two equipowered signals
with |rho|=0.50, SNR=20 dB, impinging on a 5-element ULA

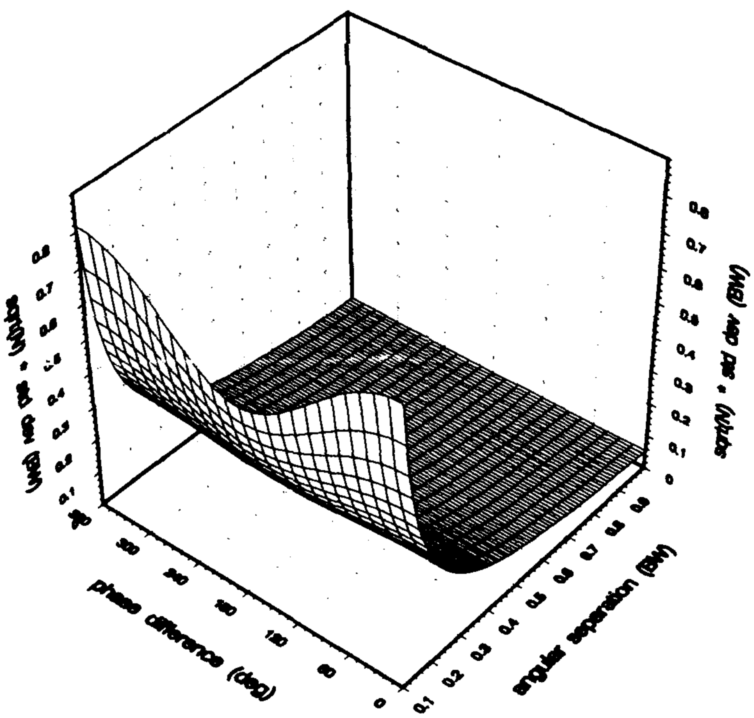

Figure 2(b). MUSIC for two equipowered signals
with |rho|=0.90, SNR=20 dB, impinging on a 5-element ULA

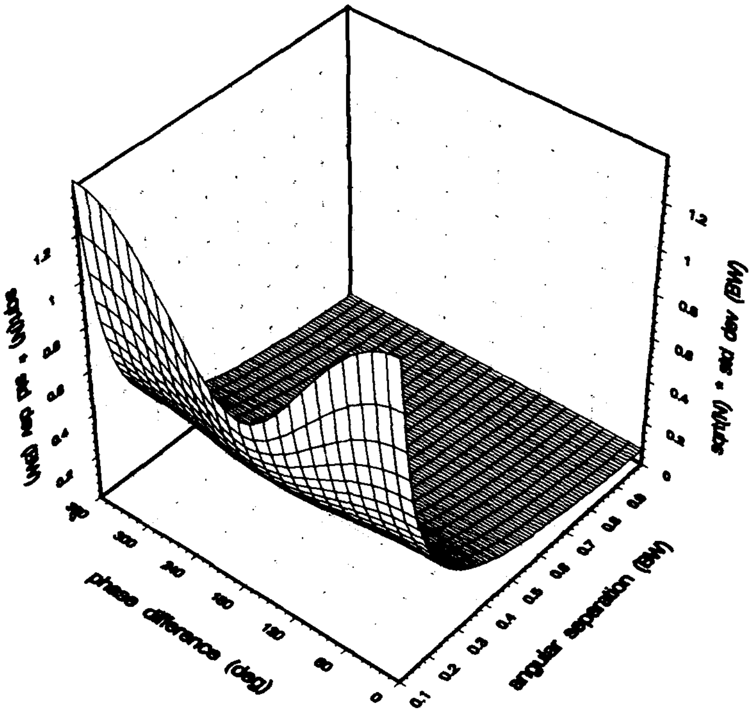

Figure 2(c). MUSIC for two equipowered signals
with |rho|=0.95, SNR=20 dB, impinging on a 5-element ULA

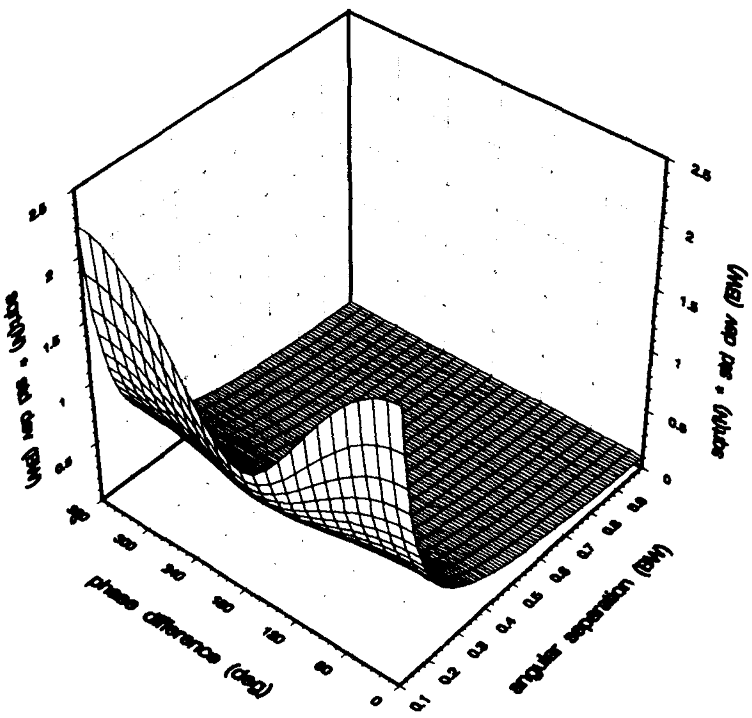

Figure 2(d). MUSIC for two equipowered signals
with |rho|=0.97, SNR=20 dB, impinging on a 5-element ULA

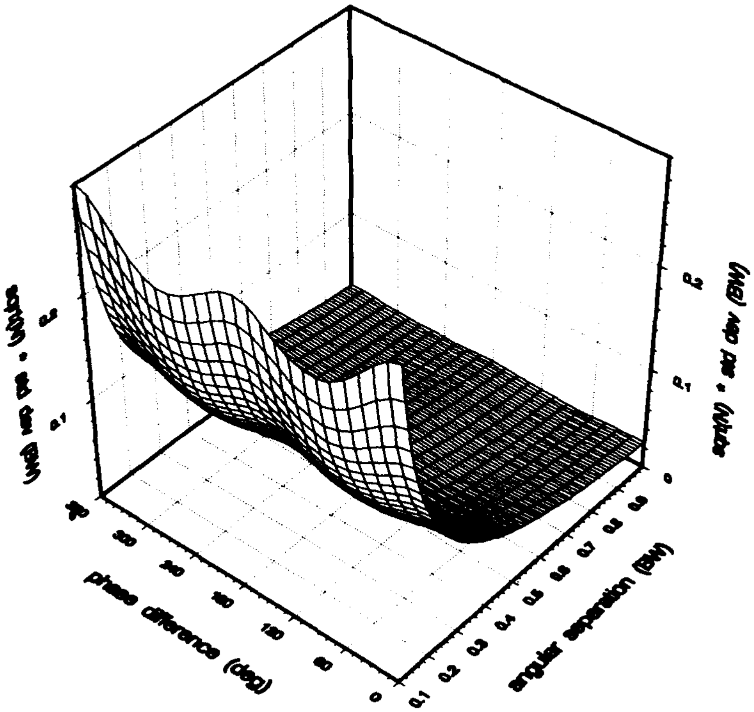

Figure 3(a). ML for two equipowered signals
with |rho|=0.50, SNR=20 dB, impinging on a 5-element ULA

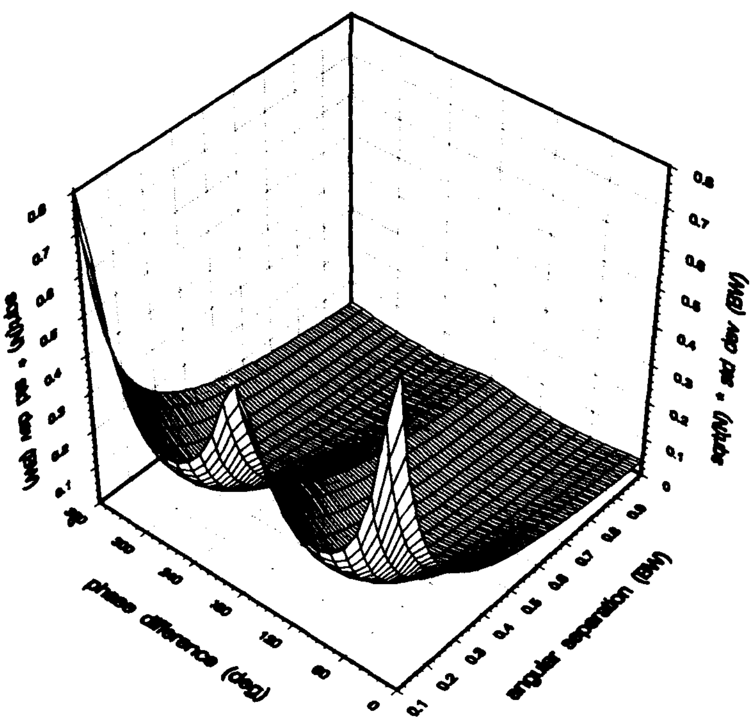

Figure 3(b). ML for two equipowered signals
with |rho|=0.90, SNR=20 dB, impinging on a 5-element ULA

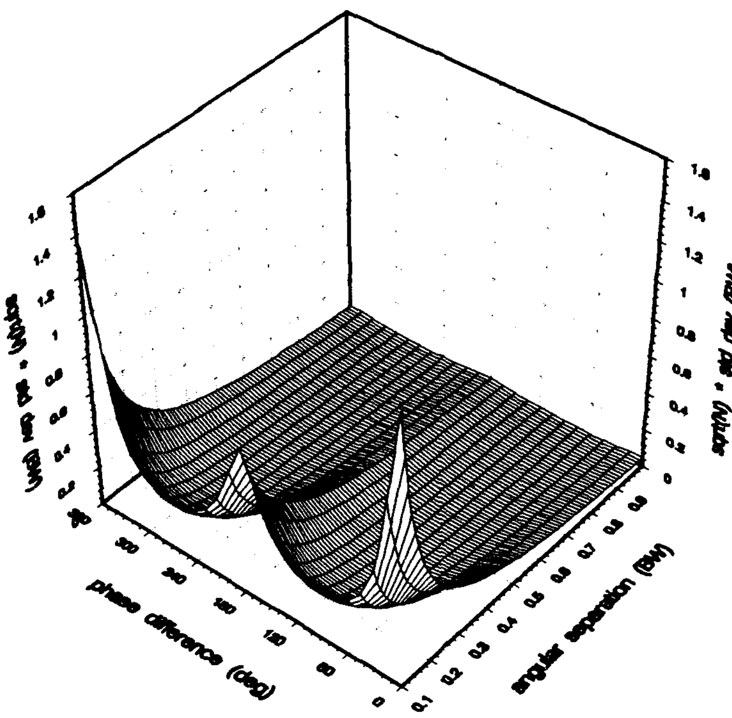

Figure 3(c). ML for two equipowered signals
with |rho|=0.95, SNR=20 dB, impinging on a 5-element ULA

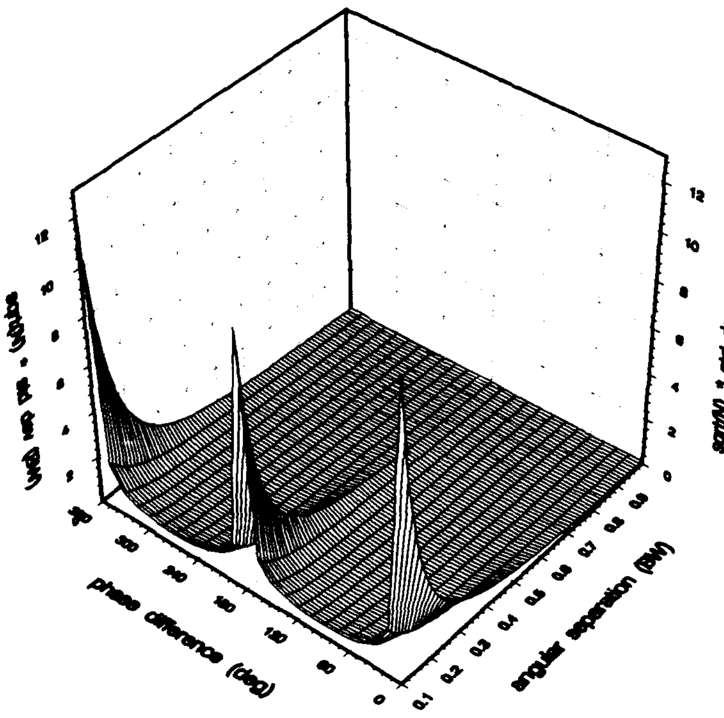

Figure 3(d). ML for two equipowered signals
with |rho|=1.00, SNR=20 dB, impinging on a ULA

Notice that the behavior of $[\mathbf{CRB}]_{ii}$, $[\mathbf{C}_{MU}]_{ii}$, and $[\mathbf{C}_{ML}]_{ii}$ with respect to the temporal phase difference θ agrees with our results (Lemma 5, and Propositions B.2 and C.2). In particular, when $\Delta\omega = 1\,\mathrm{BW}$, $[\mathbf{C}_{MU}]_{ii}$ is independent of θ in figures 2(a) to 2(d) as stated in Proposition B.2, and also when $\Delta\omega = 1\,\mathrm{BW}$, $[\mathbf{C}_{ML}]_{ii}$ behaves in the same manner as $[\mathbf{CRB}]_{ii}$ in figures 3(a) to 3(d) as expected from Proposition C.2.

Qualitatively, as discussed in sections 4 to 6, effects of temporal phase difference θ on $[\mathbf{CRB}]_{ii}$, $[\mathbf{C}_{MU}]_{ii}$ and $[\mathbf{C}_{ML}]_{ii}$ are intensified when the two signals are closely spaced and/or when their normalized correlation magnitude $|\rho|$ is high. Figures 1(a) to 3(d) exhibit this behavior.

8. CONCLUSIONS

We derive closed form expressions for the Cramér-Rao Bound, MUSIC, and ML asymptotic variances corresponding to the two-source direction-of-arrival estimation where sources are modeled as deterministic signals impinging on a uniform linear array.

The choice of the center of the array as the coordinate reference results in compact expressions that greatly facilitate our study of effects of temporal phase difference (correlation phase) of the two sources on asymptotic variances of estimation errors. For instance, the behavior of the Cramér-Rao Bound, which was obtained earlier by Evans, *et al.* (reference [15]), can now be stated, with respect to the coordinate reference at the center of the array, as follows.

(•) The CRB variance of estimation error, considered as a function of the phase difference θ, is periodic of period 180^0, symmetric about 90^0, and on the interval where $0^0 \le \theta \le 180^0$, it is decreasing when $0^0 < \theta < 90^0$, increasing when $90^0 < \theta < 180^0$, and it assumes the minimum at 90^0 and maximum (either finite or infinite) at 0^0 and 180^0 (that is, either the two signals are in phase or out of phase).

Since these effects are intensified, as reflected in derived expressions and exhibited in numerical results, when the two signals are closely spaced and/or when their normalized correlation magnitude is high, the following is a summary of our analytical results for MUSIC and ML restricted to the case when the electrical angular separation is within one beamwidth ($0 < \Delta\omega < 1\,\mathrm{BW}$).

(•) Suppose $0 < \Delta\omega < 1\,\mathrm{BW}$, then the asymptotic MUSIC variance of estimation error, considered as a function of the temporal phase difference θ, is symmetric about 180^0, decreasing when $0^0 < \theta < 180^0$, increasing when $180^0 < \theta < 360^0$, attains its maximum at $\theta = 0^0$ (that is, the two signals are in phase) and minimum at $\theta = 180^0$ (that is, the two signals are out of phase).

(•) Suppose $0 < \Delta\omega < 1\,\mathrm{BW}$, then the asymptotic ML variance of estimation error, considered as a function of the temporal phase difference θ, is symmetric about 180^0, and on the interval where $0^0 \leq \theta \leq 180^0$, it attains the maximum at $\theta = 0^0$ (that is, the two signals are in phase) and minimum at $\theta = \theta_0$ with $0^0 < \theta < \theta_0$, where the exact location of θ_0 depends on $\Delta\omega$ and other parameters (number of array elements m, normalized correlation magnitude $|\rho|$, and signal-to-noise ratios $\frac{\pi_1}{\sigma^2}$ and $\frac{\pi_2}{\sigma^2}$).

Our results provide an analytical background for simulations such as those performed in references [10] and [12].

9. REFERENCES

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APPENDIX A

Proof of Lemma 3

Although the main purpose of this appendix is to prove Lemma 3, the fundamental development can be used to show that if the two plane waves have equal powers, that is $\pi_1 = \pi_2$, then

$$[\mathbf{CRB}]_{11} = [\mathbf{CRB}]_{22}, \qquad [\mathbf{C}_{MU}]_{11} = [\mathbf{C}_{MU}]_{22}, \qquad [\mathbf{C}_{ML}]_{11} = [\mathbf{C}_{ML}]_{22}, \qquad \text{(A.1)}$$

without actually computing the corresponding variances.

To facilitate the discussion we introduce the following family of 2×2 Hermitian matrices. Let

$$\mathcal{C} = \left\{ \begin{bmatrix} a & \alpha \\ \overline{\alpha} & a \end{bmatrix} : a \ \text{real} \right\},$$

that is, $\mathcal{C}$ is the collection of 2×2 Hermitian matrices $\mathbf{M}$ with $\mathbf{M}_{11} = \mathbf{M}_{22}$. The following properties of $\mathcal{C}$ are easy to verify.

(P1): $\mathbf{M} \in \mathcal{C}$ and r real $\implies r\mathbf{M} \in \mathcal{C}$

(P2): $\mathbf{M} \in \mathcal{C} \implies \mathbf{M}^T, \mathbf{M}^*, \mathrm{Re}(\mathbf{M}) \in \mathcal{C}$

(P3): nonsingular $\mathbf{M} \in \mathcal{C} \implies \mathbf{M}^{-1} \in \mathcal{C}$

(P4): $\mathbf{M}, \mathbf{N} \in \mathcal{C} \implies \mathbf{M} + \mathbf{N} \in \mathcal{C}$

(P5): $\mathbf{M}, \mathbf{N} \in \mathcal{C} \implies \mathbf{M} \odot \mathbf{N} \in \mathcal{C}$

(P6): $\mathbf{M}, \mathbf{N} \in \mathcal{C} \implies \mathbf{MNM} \in \mathcal{C}$.

Notice that $\mathbf{MN}$ need not be a matrix in the class $\mathcal{C}$ even though $\mathbf{M}$ and $\mathbf{N}$ are members of $\mathcal{C}$.

Recall that $\bigcirc$ denotes a point between the p-th sensor and the $(p+1)$-st sensor of the array such that (10) hold for some $1 \le p \le m-1$ and $0 \le r \le 1$. Clearly, every point on the line segment connecting the first and last sensors is of the form of $\bigcirc$ for some $1 \le p \le m-1$ and some $0 \le r \le 1$.

We notice that

$$\mathbf{A}_0^*(\text{origin at } \bigcirc)\, \mathbf{A}_0(\text{origin at } \bigcirc) = \begin{bmatrix} m & \sum(\text{origin at } \bigcirc) \\ \overline{\sum(\text{origin at } \bigcirc)} & m \end{bmatrix}, \qquad \text{(A.2)}$$

where

$$\sum(\text{origin at } \bigcirc) = \sum_{k=0}^{m-1} e^{i(k-(p-1+r))\Delta\omega}$$
$$= \frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})} \cdot e^{i(\frac{m+1}{2}-(p+r))\Delta\omega}, \tag{A.3}$$

and hence

$$\det(\mathbf{A}_0^*(\text{origin at } \bigcirc)\, \mathbf{A}_0(\text{origin at } \bigcirc)) = m^2 - \left(\frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})}\right)^2,$$

which is independent of coordinate reference.

To further simplify notation we write

$$\sum\nolimits_k(\text{origin at } \bigcirc) = \sum_{k=0}^{m-1}(k-(p-1+r))e^{i(k-(p-1+r)\Delta\omega},$$
$$\sum\nolimits_{k^2}(\text{origin at } \bigcirc) = \sum_{k=0}^{m-1}(k-(p-1+r))^2 e^{i(k-(p-1+r))\Delta\omega}.$$

Then

$$\sum\nolimits_k(\text{origin at } \bigcirc) = -i\frac{d}{d\Delta\omega}\left(\sum(\text{origin at } \bigcirc)\right),$$
$$\sum\nolimits_{k^2}(\text{origin at } \bigcirc) = -i\frac{d}{d\Delta\omega}\left(\sum\nolimits_k(\text{origin at } \bigcirc)\right).$$

We are now ready to compute $\mathbf{H}$(origin at $\bigcirc$). From (16) we obtain

$$\mathbf{D}_0^*(\text{origin at } \bigcirc)\, \mathbf{D}_0(\text{origin at } \bigcirc)$$
$$= \begin{bmatrix} \sum_{k=0}^{m-1}(k-(p-1+r))^2 & \sum_{k^2}(\text{origin at } \bigcirc) \\ \overline{\sum_{k^2}(\text{origin at } \bigcirc)} & \sum_{k=0}^{m-1}(k-(p-1+r))^2 \end{bmatrix}, \tag{A.4}$$

where

$$\sum_{k=0}^{m-1}(k-(p-1+r))^2 = \frac{1}{6}m(m-1)(2m-1) + m(p-1+r)(m-p-r).$$

Next, from (13) and (16) we have

$$\mathbf{D}_0^*(\text{origin at } \bigcirc)\, \mathbf{A}_0(\text{origin at } \bigcirc) = -i\mathbf{M}_0(\text{origin at } \bigcirc), \tag{A.5}$$
$$\mathbf{A}_0^*(\text{origin at } \bigcirc)\, \mathbf{D}_0(\text{origin at } \bigcirc) = i\mathbf{M}_0(\text{origin at } \bigcirc), \tag{A.6}$$

where

$$\mathbf{M}_0(\text{origin at } \bigcirc) = \begin{bmatrix} m(m+1-2(p+r))/2 & \sum_k(\text{origin at } \bigcirc) \\ \overline{\sum_k(\text{origin at } \bigcirc)} & m(m+1-2(p+r))/2 \end{bmatrix}. \tag{A.7}$$

From (A.2), (A.4), and (A.7) we find that

$$\mathbf{A}_0^*(\text{origin at } \bigcirc)\ \mathbf{A}_0(\text{origin at } \bigcirc) \ \in \ \mathcal{C},$$

$$\mathbf{D}_0^*(\text{origin at } \bigcirc)\ \mathbf{D}_0(\text{origin at } \bigcirc) \ \in \ \mathcal{C},$$

$$\mathbf{M}_0(\text{origin at } \bigcirc) \ \in \ \mathcal{C}.$$

By the generality of the point $\bigcirc$, the matrix $\mathbf{A}_0^*\mathbf{A}_0$, $\mathbf{D}_0^*\mathbf{D}_0$, and $\mathbf{M}_0$ belong to the class $\mathcal{C}$ regardless of choice of coordinate origin.

Since

$$\begin{aligned} \mathbf{H} = \mathbf{D}_0^*\mathbf{P}_{\mathbf{A}_0}^{\perp}\mathbf{D}_0 &= \mathbf{D}_0^*\mathbf{D}_0 - \mathbf{D}_0^*\mathbf{A}_0(\mathbf{A}_0^*\mathbf{A}_0)^{-1}\mathbf{A}_0^*\mathbf{D}_0 \\ &= \mathbf{D}_0^*\mathbf{D}_0 - \mathbf{M}_0(\mathbf{A}_0^*\mathbf{A}_0)^{-1}\mathbf{M}_0, \end{aligned} \tag{A.8}$$

it follows immediately that $\mathbf{H}$ is also an element of $\mathcal{C}$ regardless of choice of coordinate origin.

Furthermore, being Hermitian, $\mathbf{H}$ is positive semidefinite. That is,

$$\mathbf{H}_{11} \geq 0; \qquad \mathbf{H}_{11}^2 - |\mathbf{H}_{12}|^2 \geq 0. \tag{A.9}$$

We claim $\mathbf{H}_{11} \neq 0$. To see this we argue as follows. If $\mathbf{H}_{11}$=0, then by the second part of (A.9) we also have $\mathbf{H}_{12}$=0 and thus

$$\mathbf{0} = \mathbf{H} \triangleq \mathbf{D}^*\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{D} = \mathbf{D}^*\left(\mathbf{P}_{\mathbf{A}}^{\perp}\right)^*\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{D}. \tag{A.10}$$

Let $\mathbf{d}_1$, and $\mathbf{d}_2$ be the two columns of $\mathbf{H}$, then (A.10) gives

$$\mathbf{d}_1^*\left(\mathbf{P}_{\mathbf{A}}^{\perp}\right)^*\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{d}_1 = 0 = \mathbf{d}_2^*\left(\mathbf{P}_{\mathbf{A}}^{\perp}\right)^*\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{d}_2,$$

$$\mathbf{d}_1^*\left(\mathbf{P}_{\mathbf{A}}^{\perp}\right)^*\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{d}_2 = 0 = \mathbf{d}_2^*\left(\mathbf{P}_{\mathbf{A}}^{\perp}\right)^*\mathbf{P}_{\mathbf{A}}^{\perp}\mathbf{d}_1,$$

which then implies $\mathbf{d}_1, \mathbf{d}_2 \in \text{span}(\mathbf{A})$. If $r = 0$ (resp., $r = 1$), then the coordinate reference is the $p-$th (resp., $(p+1)-$st) sensor element. This, however, is impossible since both column vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ have their $p-$th (resp., $(p+1)-$st) elements equal to zero while the $p-$th (resp., $(p+1)-$st) elements of $\mathbf{a}_1$ and $\mathbf{a}_2$ are both equal to one. Thus, we can suppose that $0 < r < 1$. It is clear that neither $\mathbf{d}_j$, $j = 1,2$, is a (complex) multiple of $\mathbf{a}_1$ or $\mathbf{a}_2$. Hence, if $\mathbf{d}_1 \in \text{span}(\mathbf{A})$ then there are two non-zero complex numbers α and β such that

$$i(m-(p+r)-j) = \alpha + \beta e^{i(m-(p+r)-j)\Delta\omega}, \qquad 0 \leq j \leq m-1.$$

This, however, is unattainable for $m \geq 3$. Thus, $\mathbf{H}_{11} \neq 0$, and hence $\mathbf{H}_{11} > 0$.

Finally, using (A.8) with (A.2), (A.4), and (A.7), we obtain (29) and (30) immediately. This concludes the proof of Lemma 3.

For the case of equipowered signals mentioned at the beginning of the appendix, we note that $\mathbf{S} \in C$ for $\pi_1 = \pi_2$. From (2), (4), (9) (for $0 \leq |\rho| < 1$) and (6) (for $|\rho| = 1$) we have (A.1) readily.

APPENDIX B

A Nontrivial Example

We shall consider the situation where the electrical angular separation $\Delta\omega$ is an integer multiple of (standard or first null) beamwidth, denoted by BW, with one BW equal to $2\pi/m$. That is

$$\Delta\omega = \ell \, \mathrm{BW} = \ell \left(\frac{2\pi}{m}\right).$$

Since $0 < \Delta\omega \leq \pi$ it follows that $\ell = 1, 2, ..., \lfloor \frac{m}{2} \rfloor$ where $\lfloor \frac{m}{2} \rfloor$ denotes the largest integer less than or equal to $\frac{m}{2}$.

The interesting result we derive here can be summarized as follows.

Proposition Append.B *Suppose $m \geq 4$ and $\Delta\omega = \ell$ BW, for $\ell = 1, 2, ..., \lfloor \frac{m}{2} \rfloor$. Then, for all choices of coordinate reference and any signal correlation matrix* $\mathbf{S}$, *the variance* $[\mathbf{CRB}]_{ii}$, $i = 1, 2$, *is finite.*

For $m = 3$ and $\Delta\omega = 1$ BW $= \frac{2\pi}{3}$, the variance $[\mathbf{CRB}]_{ii}$, $i = 1, 2$, *is arbitrarily large* (∞) *if and only if the two signals are correlated with appropriate phase difference θ which depends on the choice of coordinate reference.*

Proof of this proposition is carried out by the following two lemmas. It is interesting to note that, when $m = 3$, the electrical angular separation $\Delta\omega = 1\,\mathrm{BW} = \frac{2\pi}{3}$ is neither small in absolute value ($\frac{2\pi}{3}$) nor in relative value (1 BW). Physically, for interelement spacing equal to half of wavelength, two plane waves with angles of arrival (with respect to the normal to the array), say, $\phi_1 = 0^0$ and ϕ_2 such that $\sin(\phi_2) = \frac{2}{3}$ (say, $\phi_2 \approx 42^0$) give rise to this electrical angular separation.

First, using explicit expressions (29) and (30) for elements of $\mathbf{H}$ given in Lemma 3, we have immediately

$$\mathbf{H}_{11}(\text{origin at } \bigcirc) = \frac{m}{4}\left(\frac{m^2-1}{3} - \frac{1}{\sin^2\left(\frac{\ell\pi}{m}\right)}\right), \tag{B.1}$$

$$\mathbf{H}_{12}(\text{origin at } \bigcirc) = \frac{(-1)^\ell m \cos\left(\frac{\ell\pi}{m}\right)}{2\sin^2\left(\frac{\ell\pi}{m}\right)} \cdot e^{i\left(\frac{m+1}{2} - (p+r)\right)\frac{2\pi\ell}{m}}, \tag{B.2}$$

hence

$$|\mathbf{H}_{12}(\text{origin at } \bigcirc)| = \frac{m\cos\left(\frac{\ell\pi}{m}\right)}{2\sin^2\left(\frac{\ell\pi}{m}\right)}. \tag{B.3}$$

By the generality of $\bigcirc$, we conclude from (B.1) and (B.3) that $\mathbf{H}_{11}$ and $|\mathbf{H}_{12}|$ are independent of the choice of coordinate reference when $\Delta\omega = \ell$ BW.

Since the general coordinate reference is used in this appendix, to facilitate the discussion we shall not restrict the temporal phase difference to any interval; however, in all cases, by Lemma 2, we know that $[\mathbf{CRB}(\theta+\pi)]_{ii} = [\mathbf{CRB}(\theta)]_{ii}$, $i = 1,2$.

If $m = 3$ then $\ell = 1$. It follows immediately

$$\mathbf{H}_{11} = 1 = |\mathbf{H}_{12}|,$$

$$\arg(\mathbf{H}_{12}(\text{origin at } \bigcirc)) = \pi + \frac{2\pi}{3}\,(2-(p+r)).$$

By appealing to (45), we obtain

Lemma Append.B.1 *Suppose $m = 3$ and $\Delta\omega = 1$ BW $(= \frac{2\pi}{3})$. Then, for any choice of coordinate reference, the matrix* $\mathbf{H}$ *is singular. Consequently, for $m = 3$ and $\Delta\omega = 1$ BW, the variance $[\mathbf{CRB}]_{ii}$, $i = 1,2$, is arbitrary large (∞) if and only if*

$$|\rho| = 1 \quad \text{and} \quad \theta = \frac{\pi}{3}(1-2(p+r)) + k\pi, \tag{B.4}$$

where k is any integer, $|\rho|$ and θ as in (26), and p and r defined the coordinate reference $\bigcirc$ as in (10). In particular, if the center of the array is chosen as the coordinate reference, then $\theta = k\pi$ in (B.4).

We next consider the case where $m \geq 4$. For ease of notation we shall define

$$f(m;\ell) \triangleq \frac{|\mathbf{H}_{12}|}{\mathbf{H}_{11}}$$

then, by (B.1) and (B.3),

$$f(m;\ell) = \frac{2\cos\left(\frac{\ell\pi}{m}\right)}{\left(\frac{m^2-1}{3}\right)\sin^2\left(\frac{\ell\pi}{m}\right) - 1}. \tag{B.5}$$

Suppose $m \geq 4$ is fixed. Then for $1 \leq \ell < k \leq \lfloor \frac{m}{2} \rfloor$, we see that

$$\sin\left(\frac{k\pi}{m}\right) \;>\; \sin\left(\frac{\ell\pi}{m}\right) \;\; (>0),$$

and

$$(0 \leq) \ \cos\left(\frac{k\pi}{m}\right) \ < \ \cos\left(\frac{\ell\pi}{m}\right).$$

Therefore, for a fixed $m \geq 4$,

$$f(m;k) < f(m;\ell) \quad \text{for} \quad 1 \leq \ell < k \leq \left\lfloor \frac{m}{2} \right\rfloor. \tag{B.6}$$

We shall prove

$$f(m;\ell) < 1 \quad \text{for} \quad m \geq 4 \quad \text{and} \quad 1 \leq \ell \leq \left\lfloor \frac{m}{2} \right\rfloor. \tag{B.7}$$

To this end, in view of (B.6), we need only show

$$f(m;1) < 1 \quad \text{for} \quad m \geq 4. \tag{B.8}$$

From (B.5), it is clear that $f(m;1) < 1$ is equivalent to

$$1 + 2\cos\left(\frac{\pi}{m}\right) < \left(\frac{m^2-1}{3}\right) \sin^2\left(\frac{\pi}{2}\right) \quad \text{for} \quad m \geq 4. \tag{B.9}$$

We list the coresponding values of the LHS and the RHS of (B.9) for $4 \leq m \leq 7$.

m	$1 + 2cos(\frac{\pi}{m})$	$(\frac{m^2-1}{3})\sin^2(\frac{\pi}{m})$
4	$1 + \sqrt{2} \approx 2.4142$	$\frac{5}{2}$
5	≈ 2.6180	≈ 2.7639
6	$1 + \sqrt{3} \approx 2.7321$	$\frac{35}{12} \approx 2.9166$
7	≈ 2.8019	≈ 3.0121

Observe that (B.9) holds for $4 \leq m \leq 7$. Also note that the LHS of (B.9) is bounded by 3 and the RHS of (B.9) is greater than 3 for $m = 7$. Thus, to show (B.9) it suffices to show the RHS of (B.9) is an increasing function of m. To do this, we define

$$g(x) = \frac{x^2-1}{3} \sin^2\left(\frac{\pi}{x}\right), \quad x \geq 4.$$

Then

$$g'(x) = \left[\frac{x}{\pi}\sin\left(\frac{\pi}{x}\right) - \left(\frac{m^2-1}{m^2}\right)\cos\left(\frac{\pi}{x}\right)\right] \frac{2\pi}{3}\sin\left(\frac{\pi}{x}\right) \tag{B.10}$$

Using the inequality (reference [19], page 75)

$$\cos(t) \leq \frac{\sin(t)}{t}, \quad 0 \leq t \leq \pi,$$

we write

$$\begin{aligned}
\frac{x}{\pi}\sin\left(\frac{\pi}{x}\right) - \left(\frac{x^2-1}{x^2}\right)\cos\left(\frac{\pi}{x}\right) &> \cos\left(\frac{\pi}{x}\right) - \left(\frac{x^2-1}{x^2}\right)\cos\left(\frac{\pi}{x}\right) \\
&= \frac{1}{x^2}\cos\left(\frac{\pi}{x}\right) \\
&> 0, \quad \text{for } x \geq 4.
\end{aligned}$$

This proves $g'(x) > 0$ and thus $g(x)$ is an increasing function of x, for $x \geq 4$. It follows that (B.9) holds for all $m \geq 4$ which proves (B.8) which, in turn, implies (B.7).

We summarize the discussion in

Lemma Appd.B.2 *Suppose $m \geq 4$ and $\Delta\omega = \ell$ BW, for $\ell = 1, 2, \ldots, \lfloor \frac{m}{2} \rfloor$. Then the matrix* **H** *is nonsingular for all choices of coordinate reference. Consequently, the variance* $[\mathbf{CRB}]_{ii}$, $i = 1, 2$, *is finite for any choice of coordinate reference and any signal correlation matrix* **S**.

APPENDIX C

Computing $[\mathbf{C}_{MU}(\theta)]_{ii}$, $i = 1, 2$

In this appendix, we shall compute $[\mathbf{C}_{MU}(\theta)]_{ii}$, $i = 1,2$, for the center of the array as the coordinate reference.

Recall that, for $0 \leq |\rho| < 1$,

$$\mathbf{C}_{MU} = \frac{\sigma^2}{2} (\mathbf{H} \odot \mathbf{I})^{-1} \operatorname{Re}\left(\mathbf{H} \odot \mathbf{K}^T\right) (\mathbf{H} \odot \mathbf{I})^{-1},$$

where

$$\mathbf{K} = \mathbf{S}^{-1} + \sigma^2 \mathbf{S}^{-1} (\mathbf{A}^* \mathbf{A})^{-1} \mathbf{S}^{-1}.$$

Using

$$(\mathbf{H} \odot \mathbf{I})^{-1} = \left(\frac{1}{\mathbf{H}_{11}}\right) \mathbf{I},$$

together with the fact that $\mathbf{H}$ is real when the coordinate reference is chosen at the center of the array, we obtain

$$\mathbf{C}_{MU} = \frac{\sigma^2}{2\,\mathbf{H}_{11}{}^2} \mathbf{H} \odot \operatorname{Re}\left(\mathbf{K}^T\right).$$

Therefore, for $i = 1, 2$,

$$[\mathbf{C}_{MU}(\theta)]_{ii} = \frac{\sigma^2}{2\,\mathbf{H}_{11}} \left[\operatorname{Re}\left(\mathbf{K}^T\right)\right]_{ii} = \frac{\sigma^2}{2\,\mathbf{H}_{11}} [\operatorname{Re}(\mathbf{K})]_{ii} = \frac{\sigma^2}{2\,\mathbf{H}_{11}} \operatorname{Re}(\mathbf{K}_{ii}), \qquad \text{(C.1)}$$

where, by a straightforward computation,

$$\operatorname{Re}(\mathbf{K}_{ii}) = \frac{1}{\pi_i\,(1 - |\rho|^2)} \cdot \left\{ 1 + \frac{\sigma^2}{\det(\mathbf{S})} \cdot \left(\frac{1}{m^2 - \left(\frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})}\right)^2} \right) \right.$$
$$\left. \times \left[\left(\pi_j + |\rho|^2 \pi_i\right) m + 2\sqrt{\pi_1 \pi_2}\,|\rho| \cdot \left(\frac{\sin(\frac{m\Delta\omega}{2})}{\sin(\frac{\Delta\omega}{2})}\right) \cos(\theta) \right] \right\}, \quad \begin{matrix} i,j \in \{1,2\} \\ j \neq i. \end{matrix} \qquad \text{(C.2)}$$

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We derived closed form expressions for the Cramér-Rao Bound, asymptotic MUSIC, and maximum-likelihood (ML) variances corresponding to the two-source direction-of-arrival estimation where sources are modeled as deterministic signals impinging on a uniform linear array. The choice of the center of the array as the coordinate reference resulted in compact expressions that greatly facilitated our study of effects of temporal phase difference (correlation phase) of the two sources on asymptotic variances of estimation error. These effects were intensified when the two signals were closely spaced and/or when their normalized correlation magnitude was high. Numerical results obtained from specializing general formulas confirmed our results.

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